The Varieties of Indispensability Arguments

Marco Panza, Andrea Sereni

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Marco Panza
CNRS, IHPST (CNRS and Univ. of Paris 1 Panthéon-Sorbonne)
marco.panza@univ-paris1.fr

Andrea Sereni
(Institute for Advanced Studies IUSS, Pavia, NEtS Centre)
andrea.sereni@iusspavia.it

Semifinal draft. Final version published on:
Syhthese, S.I. "Indispensability and Explanation".
DOI 10.1007/s11229-015-0977-9

Abstract

The indispensability argument (IA) comes in many different versions that all reduce to a general valid schema. Providing a sound IA amounts to providing a full interpretation of the schema according to which all its premises are true. Hence, arguing whether IA is sound results in wondering whether the schema admits such an interpretation. We discuss in full details all the parameters on which the specification of the general schema may depend. In doing this, we consider how different versions of IA can be obtained, also through different specifications of the notion of indispensability. We then distinguish between schematic and genuine IA, andargue that no genuine (non-vacuously and non-circularly) sound IA is available or easily forthcoming. We then submit that this holds also in the particularly relevant case in which indispensability is conceived as explanatory indispensability.

*Many thanks to the audience of the *Indispensability and Explanation* workshop held at IHPST in Paris in November 19-20, 2012, where this paper was first presented. We are particularly grateful to Henri Galinon for his useful rejoinder at the conference and for subsequent discussion. The authors would like to thank the projects which financially supported their research for this article. Marco Panza wishes to thank the ANR-DFG Project Mathematical objectivity by representation. Andrea Sereni wishes to thank the Italian National Project PRIN 2010 Realism and Objectivity.

1 Introduction

What is today repeatedly called 'Indispensability Argument'—henceforth, IA—is, in fact, a large and multifarious family of different arguments[1]. Although most of them concern mathematics, the basic underlying idea applies to other domains. As a matter of fact, indispensability arguments exemplify a well-recognizable Quinean strategy in ontology. Suppose we have a class of theories of such a kind that we possess fairly clear means of establishing whether they are true (or justified, or confirmed, or the like) and, modulo some suitable formulation, whether the objects they purports to be about do exist (or whether belief in their existence is justified, or confirmed, or the like). Suppose that we have theories of another disputed class, a class for which we lack those probatory means, at least insofar as we look at them in isolation from their interaction with other theories. If we have a way of showing that appeal to theories of this second class cannot—in some sense to be further specified—be avoided by theories of the first class, we may have a reason to think that some appropriate semantic or epistemic property (truth, justification or confirmation being just some examples) is transferred from the former to the latter. This will give us indirect probatory means like those at our disposal for theories in the former class; this will then allow us to establish whether theories in the second class are true (or justified, or confirmed, or the like) and whether the objects they purport to be about exist (or the claim that they exist is justified, or confirmed, or the like).

Even though versions of the argument can be adapted to support claims unconcerned with ontology (e.g. that the theories in the disputed class have some suitable semantic or epistemic property quite independently of which objects they are about, and even independently of the very fact that they may be about objects at all), its role in ontology is paramount. Recently, O. Bueno and S. A. Shalkowski ([Bueno and Shalkowski., 2015], pp. 231-32) have thus summarized a typical application of the argument when the main concern is ontology[2]:

[1]In what follows, we shall distinguish between genuine (or fully determinate) arguments and argument schemas. We shall generally speak of arguments *tout court* in order to refer either to the former or to the latter or to both at once.

[2]As Bueno and Shalkowski make clear, "the indispensability argument *per se* does not establish the nature of [...][the relevant] objects (whether such objects are contingent or not) nor does it establish the form of knowledge one may have [...] (whether such knowledge is *a priori* or not). The argument [...] only establishes that we ought to be ontologically committed to [...][these] objects. This leaves open the possibility of using indispensability considerations in support of an ontology of non-contingent objects whose knowledge is justified on *a priori* grounds." As we also have stressed somewhere else (cf. [Panza and Sereni, 2015]), although the indispensability argument, being based on premises that may be established on *a posteriori* grounds—such as the form that proper formulations of scientific theories must take in order to be suitable for description, explanation or prediction—is usually a powerful tool in the hands of empiricists willing to defend the existence of mathematical abstract objects on empirical grounds, the argument *per se* does not rule out the availability of other *a priori* reasons for believing in those objects—in which case, it will at most offer some auxiliary, defeasible, and less than ideal evidence for the wanted conclusion

> *Indispensability considerations*: Positing certain objects is sometimes thought to be indispensable to express certain claims about the world, or to provide a systematic description of a certain range of phenomena. [...] Hence, we are ontologically committed to such objects. [...] a weaker form of the indispensability argument can be formulated as a theoretical utility argument. In this version of the argument, the fact that reference to a certain class of objects, is theoretically useful provides reason to believe in such objects. Simplicity, unification, expressive and explanatory power are all virtues that are commonly invoked in support of the belief in the objects in question.

This should be enough to show that IA does not need to be confined to enquiries about mathematics. And in fact it isn't. Examples could be multiplied. One is offered by M. Colyvan ([Colyvan, 2001], p. 16) who, following a suggestion from S. J. Wagner ([Wagner, 1996], p. 76), has sketched a "semantic indispensability argument", based on the indispensability of the "apparent reference" to abstract entities like predicates and logical operators for our "best semantic theories"[3]. Another is given by [Enoch and Schechter, 2008], where the appeal to the indispensability of some basic forms of inference to some unavoidable rational projects is used in order to claim that thinkers are justified in adopting those forms of inference[4].

In the present paper, we shall not be concerned with such a variety of arguments, however. We shall rather focus on the forms that IA takes when it is concerned with mathematics. For the sake of simplicity, this restriction of context will be left implicit throughout, unless required: we shall merely speak of IA or IA's to refer to arguments concerned with mathematics. One of our main theses will indeed be that also when it is so restricted, IA displays quite a large variety of possible forms. We shall suggest ways of classifying these forms and of dealing with all or several of them at once.

Let us start by observing that, although all these forms include a crucial premise roughly stating that mathematics plays an indispensable role in science, their ultimate purpose may vary from case to case. As a consequence, their conclusion varies too, though it always aims at stating either a form of ontological mathematical platonism (as we shall call it), this being roughly the thesis that there exist objects that mathematics is about[5], or a form of mathematical veridicalism

[3]Something akin to a form of indispensability argument for abstract entities in semantics can be traced back to [Church, 1951]. Thanks to an anonymous reviewer for pointing us to Church's classic paper in this connection. As noticed by [Psillos, 1999], pp.10-11, the indispensability of the use of theoretical terms in the formulation of "efficacious" systems of laws is claimed in [Carnap, 1939], p.64.

[4]Thanks to Maria Paola Sforza Fogliani for bringing this to our attention.

[5]This thesis (or similar ones) is usually identified with mathematical platonism *tout court*. We take it to stand for ontological mathematical platonism for the reason that we take mathematical platonism *tout court* to stand for the more general thesis that mathematical statements are about some objects. It seems possible to argue for this latter claim without arguing for the existence of these objects (at least if the notion of existence is considered,

(as we shall call it), this being roughly the thesis that mathematical theorems, or other appropriate mathematical statements, are true[6].

This variety of arguments is often acknowledged, albeit implicitly, also in the limited case of mathematics. Colyvan speaks for instance of 'indispensability arguments' in the plural in order to identify the subject-matter of his influential book, in which several and quite different examples are discussed ([Colyvan, 2001], esp. § 1.2). More recently, the label 'Enhanced Indispensability Argument' coined by A. Baker to refer to his well-know argument (cf. page §3.5, below), suggests the availability of yet different IA's. To the best of our knowledge, however, before we tentatively tackled the question in [Panza and Sereni, 2013], (ch. 6) and [Panza and Sereni, 2015], no thorough investigation had been pursued in order to identify and clarify both the traits common to all possible IA's (or to IA itself, considered as a whole family of arguments), those features that are specific to each of them, and the way a single such argument may be obtained through the specification of a general pattern.

In those occasions, we suggested that the different IA's available in the current debate could be traced back to four argument schemas, respectively identifying arguments for platonism and veridicalism[7], either in epistemic or non-epistemic form—a non-epistemic argument for p being conceived as an argument whose conclusion is just p, and an epistemic one as an argument whose conclusion is that we are justified in believing p (or in taking p to be true, or generally to hold), or that p is justified (or that p is to be justifiably taken to be true, or generally to hold)[8].

Our present purpose is twofold: on the one hand, we want to come back to this issue in more details, also by discussing soundness conditions for IA's, including those which are based on the explanatory role of mathematics; on the other hand, we want to offer evidence in support of the claim that no genuine (non-vacuously and non-circularly) sound IA is currently available, and at

as it usually is, as an universal and primitive notion already clear in itself). This issue goes beyond the scope of the present paper, however.

[6]This thesis is often referred to by labels such as 'mathematical realism', 'semantic realism in mathematics', or similar ones. We adopted this convention in [Panza and Sereni, 2013], but we opt here for 'mathematical veridicalism' in order to avoid confusion with other theses often referred to as 'realism', such as the thesis that mathematical statements have a determinate truth value independently of any sort of justification one may have for them. On some occasions, the bare term 'realism' is used to refer to ontological realism, i.e. platonism, e.g. in [Field, 1982]. We avoid this use. . We will later identify another possible conclusion for IAs beyond ontological mathematical platonism and veridicalism, which will be labeled externality; cf. Sect. 3.6.

[7]Cf. footnote 6, above.

[8]We shall prefer the latter passive formulation ('p is justified') to the active one ('We are justified in believing in p'), but we do not mean to attach any particular relevance here to the distinction between doxastic and propositional justification. Notice that the conclusions of many IA's are stated in terms of ought's: namely, that we ought to do something, e.g. to believe in the relevant thesis, or to be commited to the relevant objects. For our present purpose, we just assume that if we are justified in believing a given thesis, we ought to believe it, that we ought to believe it just in case we are justified in believing it, and that being committed to some objects is the same as believing that they exist.

the same time raise doubts that any may be forthcoming – cf. §2 and Footnote 10 for clarifications of the notions of genuine IA, and non-circular and non-vacuous soundness. Our discussion will then by necessity be conducted at a certain degree of generality: in broad terms, we will point to those general schemas that different versions of IA exemplify, and we will motivate our skepticism that (non-vacuously and non-circularly) sound instances of those schemas can really be offered. We have however made explicit, wherever possible, all connections with the relevant aspects of the current debate and with the most common versions of IA.

The discussion will be divided into three main sections. In section 2, we introduce the distinction between genuine and schematic IA's. In section 3, we focus on several aspects of schematic IA's: we offer a general argument schema from which several different formulations can be obtained via the specification of various parameters; we introduce the distinction between proper and strengthened IA's; we discuss the distinction between ontological and epistemic IA's; we offer a detailed analysis of what we take the notion of indispensability to come to, and of what we take to be indispensable to what; and, finally, we distinguish between IA's with different conclusions. In section 4, we then focus on the possibility of obtaining genuine (non-circularly) sound IA's from specifications of the various parameters of the main argument schema: we first consider the features of the strengthened IA's, and then discuss at length the possibility of obtaining genuine (non-circularly) sound IA's (proper or strengthened) based either on descriptive and predictive indispensability, or, eventually, on explanatory indispensability. We finally offer some concluding remarks to summarize the outcome of our discussion and to sustain our skepticism that any genuine (non-circularly) sound IA is available or forthcoming.

To help the reader follow our arguments without too much distress for the amount of technical details, at the beginning of each section and subsection we offer general and rather relaxed digests (typographically signaled by indented paragraphs), in which the main points of the following arguments are summarized.

2 Schematic and Genuine IA's

Section summary. We distinguish between schematic and genuine IA's. We suggest that any single IA results from successive specifications of a unique pattern provided by a general argument schema, $\mathbf{Sc.IA}_0$. Specifications of $\mathbf{Sc.IA}_0$ proceed through progressive specifications of its indeterminate parameters with progressively more determinate constants (in a sense to be clarified). Genuine IA's are those in which all indeterminate parameters are replaced by fully determinate constants. We discuss validity and soundness for both schematic and genuine IA's. We suggest that, apart from trivial or inadmissible cases, $\mathbf{Sc.IA}_0$ is such that it cannot be claimed to be sound or unsound by itself, but only when some appropriate specification of it is offered.

We suggest that any single indispensability argument should be considered as resulting from successive specifications of a unique pattern, provided by a general argument schema, which we call '$\mathbf{Sc.IA}_0$'[9]. By this we mean that this pattern consists in a system of formulas (namely three premises and one conclusion) including some indeterminate parameters, together with some replacement clauses. Passing from this general schema to a single determinate IA requires replacing all the indeterminate parameters with appropriate, fully determinate constants (namely constants whose intended meaning has been fixed precisely, or at least as clearly as it is required to avoid ambiguities within the relevant community of experts), according to these clauses. We shall call any single argument obtained in this way 'genuine IA'. A genuine IA will thus be a system of statements (three premises and one conclusion) involving (together with logical constants and bound variables, if any) only fully determinate constants. It follows that, even once the validity of a schematic IA is granted, the truth of the premises of a genuine IA, and thus the soundness of the latter, will hinge on the availability of suitable replacements for all the indeterminate parameters in the corresponding premises of $\mathsf{Sc.IA}_0$ with such constants (this shall become clearer later).

We shall describe the passage from $\mathbf{Sc.IA}_0$ to a genuine IA as a process involving different steps: at each step the indeterminate parameters which occur in the argument schema at the previous step, or some of them, are replaced with other, less indeterminate parameters, or with fully determinate constants. We take an indeterminate parameter to be less indeterminate than another if the latter admits of all replacements admitted by the former, but not *vice versa*. It follows that, with the only exception of the final step resulting in a genuine IA, any such step transforms an argument schema into another, less indeterminate, argument schema. We shall call any such argument schema 'schematic IA'. $\mathbf{Sc.IA}_0$ is then a schematic IA, and any genuine IA is to be regarded as an instance of it. It can be seen as the single root of a tree, whose final outcomes are different genuine IA's and whose intermediate nodes are schematic IA's, all less indeterminate than $\mathbf{Sc.IA}_0$. The whole process by which the entire tree is obtained can then be viewed as resting on alternative progressive determinations of the indeterminate parameters.

We take argument schemas to be either valid or invalid (according to an underlying logic). $\mathbf{Sc.IA}_0$ is valid (according to a wide selection of logics, including classical and intuitionist ones), because the associated replacement clauses ensure that its indeterminate parameters admit of the same replacement instances in any of their occurrences in the schema, and, once this condition is satisfied, the schema's conclusion follows logically from the premises (according to the underlying logic). The way other schematic and genuine IA's are obtained from the schema also ensures that they are all valid (according to the same logics).

Things stands differently with soundness.

We can say that a valid argument schema is sound or unsound as such, only if it is such that either all of its premises are true, or at least one of them is untrue, under any licensed replacement of their indeterminate parameters with fully determinate constants. This can happen

[9]But cf. footnote (14), below.

in three distinct cases. The first holds independently of which particular replacement clauses are admitted: it can just be the case that all the premises of an argument schema are true under any replacement of their indeterminate parameters with any fully determinate constant of the appropriate syntactical type, or that some of these premises are untrue under any such replacement. The second and the third case depend on the admitted replacement clauses instead. The second case occurs when these clauses entail that all the premises of the schema are true under any licensed replacement of their indeterminate parameters with fully determinate constants, or that some of them are untrue under any such replacement. The third case occurs when the replacement clauses are such that—though no single one of them is untrue under any licensed replacement of their indeterminate parameters with constants—these premises are never all true at once under any such replacement. **Sc.IA**$_0$ falls neither under the first nor under the second case. One of the main tasks we shall pursue in what follows is to investigate whether it falls under the third one.

A genuine IA is of course sound if all of its premises are true. Hence, the problem is to see whether it is possible to obtain a sound genuine IA from **Sc.IA**$_0$ by the full determination of its indeterminate parameters. In § 4 we shall argue that this is far from easy, if one wants to avoid both the trivial case in which the argument is vacuously sound, and the inadmissible case in which it is circular[10]. This suggests that once the determinations that would make **Sc.IA**$_0$ vacuously sound or circular are ruled out, the schema falls under the third among the previous cases.

To the best of our knowledge, no single indispensably argument advanced up to now rules out this possibility. The reason is not that any such argument is unsound, vacuously sound or circular, but rather that none of them is, as a matter of fact, a genuine IA. As a matter of fact, all of them are schematic IA's (falling neither under the first nor under the second cases), although all certainly more determined than **Sc.IA**$_0$.

3 Schematic IA's

3.1 The General Argument Schema

Section summary. This section gives a formulation of the general argument schema for IA that was introduced in the previous section. The argument is offered both in natural language (**Sc.IA**$_0$) and in a first-order formal language (***Sc.IA**$_0$). The formulations display the parameters whose specification is required in order to obtain genuine IA's.

[10] We call an argument 'vacuously sound' if all its premises are true, but at least one of them is vacuously so. We call it 'circular' if one of its premises cannot be argued for if its conclusion is not previously granted. A circular argument can of course be sound: this happens if all its premises (and then also the conclusion) are true. In this case, it is said to be 'circularly sound'. We shall give examples in § 4.

It is also observed that any such argument delivers merely sufficient conditions for its conclusion.

Let us begin by presenting **Sc.IA**$_0$. It goes as follows:

[**Sc.IA**$_0$]

i) Some scientific theories are P.

ii) Among them, some are such that some mathematical Q's are $\mathfrak{L}$-indispensable to them.

iii) If some Q's are $\mathfrak{L}$-indispensable to some scientific theories which are P, then their a's meet the condition $\mathcal{A}$.

Hence

iv) The a's of some mathematical Q's meet the condition $\mathcal{A}$.

This schema can easily be formalized in first-order predicate language:

[***Sc.IA**$_0$]

i) $\exists x\,[\textsc{ScTh}\,(x) \wedge P\,(x)]$

ii) $\exists x\,[\textsc{ScTh}\,(x) \wedge P\,(x) \wedge \exists y\,[Q\,(y) \wedge \textsc{Math}\,(y) \wedge \mathfrak{L}\text{-}\textsc{Ind}\,(y,x)]]$.

iii) $\forall y\,[Q\,(y) \Rightarrow [\exists x\,[\textsc{ScTh}\,(x) \wedge P\,(x) \wedge \mathfrak{L}\text{-}\textsc{Ind}\,(y,x)] \Rightarrow \mathcal{A}\,(a_y)]]$.

Hence

iv) $\exists y\,[Q\,(y) \wedge \textsc{Math}\,(y) \wedge \mathcal{A}\,(a_y)]$

The existential quantification in ***Sc.IA**$_0$ should not be taken as contributing to an explanation of the meaning of the premises (*i*) and (*ii*) of **Sc.IA**$_0$. It is rather the intended meaning of these premises that should shed light on how the existential quantifiers involved in ***Sc.IA**$_0$ are to be understood. It remains, however, that ***Sc.IA**$_0$ displays the logical form of **Sc.IA**$_0$.

For one thing, it shows that premises (*i*) and (*ii*) are not independent of each other (since the latter entails the former). Distinguishing them is just a way of emphasizing the crucial role played by the assumption that some scientific theories have an appropriate property designated by 'P'. ***Sc.IA**$_0$ also makes clear that the word 'some' in premise (*iii*) is not intended to restrict the range of the implication, but is rather part of the pluralisation of 'Q' (that is, 'If some Q's are ... then their a's ...' is not to be read as 'For some Q's, it happens that if they are ... then its a's ...', but as 'It happens that if some Q's are ... then its a's ...')[11]. Furthermore, ***Sc.IA**$_0$ makes clear that the replacement clauses relative to parameter 'a' have to specify the relations linking the relevant Q's and a's (this is displayed by the rendering of this parameter through a functional constant). Finally, ***Sc.IA**$_0$ makes clear that, beyond the indeterminate parameters 'P', 'Q', '$\mathfrak{L}$', 'a' and '$\mathcal{A}$', **Sc.IA**$_0$ involves also three non-logical constants: 'scientific theories',

[11]This use of 'some' in premise (*iii*) of **Sc.IA**$_0$ might appear odd. We shall discuss the reason for adopting it in § 3.5.

'mathematical', and 'indispensable'. Fixing the meaning of $\textsf{\textbf{Sc.IA}}_0$ requires, then, both fixing the replacement clauses relative to these parameters, and fixing the meaning to be assigned to these non-logical constants.

Conclusion (iv) of $\textsf{\textbf{Sc.IA}}_0$ (as well as $^*\textsf{\textbf{Sc.IA}}_0$) does not by itself display whether the conditions expressed in the premises of the argument offer merely sufficient or also necessary conditions for the relevant a's sto meet the condition $\mathcal{A}$. It is however important to notice that $\textsf{\textbf{Sc.IA}}_0$ (as well as $^*\textsf{\textbf{Sc.IA}}_0$) only offers sufficient conditions for its conclusion. As regards the specification of the a's of some mathematical Q's which meet the condition $\mathcal{A}$, all that $\textsf{\textbf{Sc.IA}}_0$ and $^*\textsf{\textbf{Sc.IA}}_0$ say is, indeed, this:

[$\textsf{\textbf{Sc.IA}}_0$] *Suf*) In order for the a's of some mathematical Q's to meet the condition $\mathcal{A}$, it is sufficient that these Q's be $\mathfrak{L}$-indispensable to some scientific theories which are P.

[$^*\textsf{\textbf{Sc.IA}}_0$] *Suf*) $\forall z \forall x \left[\left[\begin{array}{l} Q(z) \wedge \textsc{Math}(z) \wedge \\ \textsc{ScTh}(x) \wedge P(x) \wedge \mathfrak{L}\text{-}\textsc{Ind}(z,x) \end{array} \right] \Rightarrow \mathcal{A}(a_z) \right]$.

As a matter of fact, this is nothing but a slight weakening of premise (iii)[12]. Hence, though nothing would forbid adding it to the argument's conclusion as an additional conjunct in (iv), this would in no way make this conclusion in itself stronger or more perspicuous. It would however make clear that the argument does not imply that the a's of mathematical Q's which are $\mathfrak{L}$-indispensable to some scientific theories which are P are the only a's of some Q's that meet condition $\mathcal{A}$: as already said, the argument only gives sufficient conditions for these a's to meet this condition. This may not be the case for all possible formulations of IA, as we will see in the next section.

3.2 Strengthened IA's

Section summary. A strengthened version of schematic IA which delivers not just sufficient but also necessary conditions for its conclusion is presented and discussed, and distinguished from proper schematic IA, delivering only sufficient conditions. The strenghtened argument is given both in natural language ($\textsf{\textbf{Sc.SIA}}_0$) and in first-order language ($\textsf{\textbf{*Sc.SIA}}_0$). An additional clause expressing the required necessary conditions is added to its conclusion. Although controversial, strengthened IA's may be used to account for the role of naturalism in the debate on IA.

Some may want to offer a version of IA offering not merely sufficient but also necessary conditions for the a's of some mathematical Q's to meet the condition $\mathcal{A}$, i.e. an argument for the

[12]This becomes evident when $^*\textsf{\textbf{Sc.SIA}}_0.iii$ is written in prenex normal form.

claim that the fact that Q's are $\mathfrak{L}$-indispensable to some scientific theories which are P is both sufficient and necessary for the relevant a's to meet condition $\mathcal{A}$. This is related to the alleged role that the assumption of some form of naturalism concerning mathematics is meant to play in some formulations of IA (such as that offered by [Colyvan, 2001], p. 11). When it is adapted to the present setting, this is the form of naturalism which is involved in the claim that the a's of some mathematical Q's meet the condition $\mathcal{A}$ only if these Q's are $\mathfrak{L}$-indispensable to some scientific theories which are P.

Once the distinction between merely sufficient and also necessary conditions offered by IA is brought to the fore, it is easy to state a new argument schema, essentially different from $\mathbf{Sc.IA}_0$, let us say $\mathbf{Sc.SIA}_0$ (reasons for this label will be clear soon), which delivers necessary conditions for the a's of some mathematical Q's to meet the condition $\mathcal{A}$. The first two premisses of this new argument schema are just the same as those of $\mathbf{Sc.IA}_0$. The third premise is like that of $\mathbf{Sc.IA}_0$, except for the fact that the if-then implication is replaced by a if-and-only-if implication. In place of premise $\mathrm{Sc.IA}_0.iii$, $\mathbf{Sc.SIA}_0$ and $^*\mathbf{Sc.SIA}_0$ include then the following premise:

[$\mathbf{Sc.SIA}_0$] *iii*) The a's of some Q's meet the condition $\mathcal{A}$, if and only if these Q's are $\mathfrak{L}$-indispensable to some scientific theories which are P.

$$[^*\mathbf{Sc.SIA}_0] \quad iii)\, \forall y \left[Q(y) \Rightarrow \left[\exists x \left[\textsc{ScTh}(x) \wedge P(x) \wedge \mathfrak{L}\text{-}\textsc{Ind}(y,x)\right] \Leftrightarrow \mathcal{A}(a_y)\right]\right].$$

Making this change does not affect, of course, the possibility of deriving conclusions $\mathbf{Sc.SIA}_0.iv$ and $^*\mathbf{Sc.SIA}_0.iv$. But it allows to better specify the a's of some mathematical Q's which meet the condition $\mathcal{A}$. As regards to that, $\mathbf{Sc.SIA}_0$ and $^*\mathbf{Sc.SIA}_0$ claim indeed for more than $\mathbf{Sc.IA}_0$ and $^*\mathbf{Sc.IA}_0$, namely that:

[$\mathbf{Sc.IA}_0$] *Nec-Suf*) In order for the a's of some mathematical Q's to meet the condition $\mathcal{A}$, it is necessary and sufficient that these Q's be $\mathfrak{L}$-indispensable to some scientific theories which are P.

$$[^*\mathbf{Sc.SIA}_0] \quad \textit{Nec-Suf}) \quad \forall z \forall x \left[\left[\begin{array}{l} Q(z) \wedge \textsc{Math}(z) \wedge \\ \textsc{ScTh}(x) \wedge P(x) \wedge \mathfrak{L}\text{-}\textsc{Ind}(z,x) \end{array}\right] \Leftrightarrow \mathcal{A}(a_z)\right].$$

In a way, the situation here parallels that of $\mathbf{Sc.IA}_0$ and $^*\mathbf{Sc.IA}_0$; these are nothing but slight variants of the new premises $\mathbf{Sc.SIA}_0.iii$ and $^*\mathbf{Sc.SIA}_0.iii$. Still, including them in the conclusion has now two clear related advantages: it makes the right-to-left implication in these premisses relevant (which is not the case, *mutatis mutandis*, in Colyvan's formulation of the argument, as he openly acknowledges: [Colyvan, 2001], p.12); and it makes manifest the additional strength of premises $\mathbf{Sc.SIA}_0.iii$ and $^*\mathbf{Sc.SIA}_0.iii$ over premises $\mathbf{Sc.IA}_0.iii$ and $^*\mathbf{Sc.IA}_0.iii$, by making evident that, according to the new argument, only the a's of mathematical theories Q considered in premise (*ii*) both of $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$ meet condition $\mathcal{A}$.

With this addition, the conclusions of $\mathbf{Sc.SIA}_0$ and $^*\mathbf{Sc.SIA}_0$ take the following form:

[$\mathsf{Sc.SIA}_0$] *iv)* The a's of some mathematical Q's meet the conditions $\mathcal{A}$; namely, this is so for the a's of all mathematical Q's which are $\mathfrak{L}$-indispensable to some scientific theories which are P, and only for them.

$$[{}^*\mathsf{Sc.SIA}_0] \quad iv) \quad \begin{array}{l} \exists y \left[Q\left(y\right) \wedge \textsc{Math}\left(y\right) \wedge \mathcal{A}\left(a_y\right)\right] \wedge \\ \forall z \forall x \left[\left[\begin{array}{l} Q\left(z\right) \wedge \textsc{Math}\left(z\right) \wedge \\ \textsc{ScTh}\left(x\right) \wedge P\left(x\right) \wedge \mathfrak{L}\text{-}\textsc{Ind}\left(z,x\right) \end{array} \right] \Leftrightarrow \mathcal{A}\left(a_z\right) \right]. \end{array}$$

Many discussions of IA's make sense only if related to arguments with the logical structure of $\mathsf{Sc.SIA}_0$. This is clear when the allegedly essential role of naturalism (roughly understood as said above) in IA is discussed. Moreover, once appropriately determined, $\mathsf{Sc.SIA}_0$ seems to square quite nicely with Quine's views.

Still, an argument with the logical structure of $\mathsf{Sc.SIA}_0$ is too strong if one wishes such an argument to respect the logical structure of the one first explicitly advanced by Putnam in 1971[13]. For this delivers only a sufficient condition for the relevant mathematical items to be as it claims they to be. $\mathsf{Sc.SIA}_0$, as well as any argument issued from it, should, then, be intended as strengthened IA's, and taken to differ essentially from proper ones[14].

3.3 Ontological and Epistemic IA's

Section summary. We begin the discussion of various possibilities of replacements for indeterminate parameters in the schematic arguments $\mathsf{Sc.IA}_0$ and $\mathsf{Sc.SIA}_0$. In particular, several understandings of 'scientific theories' and of parameter 'P' are discussed. A broad distinction between ontological and epistemic argument is introduced. The former are obtained by replacing 'P' by predicates standing for properties which scientific theories may possess only thanks to what they say of the world and how the worlds actually is (e.g. truth, approximate truth, truth-likeness, high objective probability). The latter are obtained by replacing 'P' by predicates standing for properties which scientific theories may possess because of our epistemic attitudes towards them (e.g. rationally justification, (empirical) confirmation, corroboration, high subjective

[13] Cf. [Putnam, 1971], p. 347:

> So far I have been developing an argument for realism roughly along the following lines: quantification over mathematical entities is indispensable for science, both formal and physical, therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine [...]

We discuss the role of naturalism with respect to IA in [Panza and Sereni, 2015].

[14] In what follows we shall generically speak of IA's either to refer both to proper and strengthened IA's at once, or in contexts where it is clear that the reference is to proper ones. We shall never specifically refer to strengthened IA's as IA's *tout court*.

probability). Less indeterminate parameters, namely 'P^O' and 'P^E', respectively, are introduced to distinguish these two sorts of arguments.

Let us come now to the constants and parameters involved in $\mathsf{\textbf{Sc.IA}}_0$ and $\mathsf{\textbf{Sc.SIA}}_0$.

Let us begin with 'scientific theories'. Though not all the IA's currently discussed explicitly include this term, the reference to scientific theories is always there, possibly implicitly. In Putnam's original formulation[15] for example, what is at stake is the indispensability of the "quantification over mathematical entities [...] for science", but, clearly, 'science' is here meant to refer to some (or possibly all) scientific theories.

What should we mean by 'scientific theories'? A proper answer to this question is often either ignored or simply presupposed in this context. In the (original) spirit of the argument, it will be fair to assume that that term refers to actual construals widely admitted as being part of some empirical science, this being conceived as some sort of enquiry (whose features can then be further specified) resulting from some sort of empirical experimental practice[16].

This pre-empts an important question: is mathematics to be considered as (part of) science? The question whether it is (or better whether there is an appropriate meaning that could be plausibly ascribed to the term 'science' according to which this is so, or whether mathematics is continuous to what is usually and uncontroversially named 'science') is a complex and vexed one. In the context of debates on IA, it has repeatedly been tackled by Penelope Maddy, who criticizes Quine for unduly restricting the scope of science to natural sciences. For example, in [Maddy, 1997, p. 184] she claims that:

> To judge mathematical methods from any vantage-point outside mathematics, say from the vantage-point of physics, seems to me to run counter to the fundamental spirit that underlies all naturalism [...] What I propose here is a mathematical naturalism that extends the same respect to mathematical practice that the Quinean naturalist extends to scientific practice. It is, after all, those methods—the actual methods of mathematics—not the Quinean replacements, that have led to the remarkable successes of modern mathematics. Where Quine holds that science is 'not

[15] Cf. footnote (13), above.

[16] The recent discussion on IA's usually places these arguments in a scientific realist framework, where scientific theories are roughly conceived as literally, or at least approximately (if literally understood), true descriptions of some external reality. This can be rendered in our schematic arguments by choosing an appropriate substitution for 'P', as well as by giving some appropriate further characterization of the features of scientific theories. It must certainly be acknowledged that some of the views of earlier proponents of IA, like Quine and Putnam, can be seen as conflicting with a scientific realist position (e.g. Quine's views on ontological relativity or Putnam's views on internal realism). And, indeed, appropriate determinations of the parameters in our schemas, apt to reflect appropriate conceptions of what scientific theories are intended to be, could also deliver genuine IA's compatible with those non-realist views. We thank an anonymous reviewer for prompting a clarification of this point.

answerable to any supra-scientific tribunal' [...] the mathematical naturalist adds that mathematics is not answerable to any extra-mathematical tribunal and not in need of any justification beyond proof and the axiomatic method[17].

A proper investigation of this issue would take us far afield. Here, taking 'scientific theories' to refer only to construals belonging to some empirical science should not be intended, of course, as implying any philosophical pronouncement to the effect that mathematics or logic or any other branch of knowledge which is usually considered not empirical (or *a priori*) is not scientific in some appropriate sense. We shall merely conform to a current habit, generally adopted within the discussion about IA, according to which 'science' and its cognates are taken to refer only to empirical disciplines, by considering it as nothing but a convenient terminological stipulation.

Once this is made clear, it is important to notice that from the reading of 'scientific theory' suggested above it follows that the parameter 'P' in our schemas is to be replaced with a monadic predicate designating a property that can be attributed to scientific theories conceived as above. Claiming that some scientific theories have this property thus comes down to claiming some among the actual construals widely admitted as being part of an empirical science posses this property. The existential import of this claim is then quite weak[18].

This does not by itself make the claim a weak one. Its strength depends, indeed, on the property designated by the predicate replacing 'P'. Two distinct roads open up here, leading to two quite different sorts of arguments. This property can either be an ontological property of scientific theories, that is a property that these theories are supposed to have merely on the basis of what they say about the empirical world, and how the world actually is (thus independently of our epistemic attitudes towards these theories); or an epistemic property of these theories, i.e. a property they are supposed to have because of our epistemic attitude towards them. Arguments we encounter along the first road will be called 'ontological', those we encounter along the second will be called 'epistemic'.[19]. This distinction could be displayed by replacing 'P' in $\mathbf{Sc.IA}_0$ respectively with the less indeterminate parameters 'P^O' and 'P^E', which provide two less indeterminate schematic IA's: $\mathbf{Sc.IA}_1^O$ and $\mathbf{Sc.IA}_1^E$, respectively. The same holds, of course, for $\mathbf{Sc.SIA}_0$, from which $\mathbf{Sc.SIA}_1^O$ and $\mathbf{Sc.SIA}_1^E$ can respectively derive[20].

[17] Maddy has further developed her views in [Maddy, 1997] and [Maddy, 2011]

[18] Some may suppose this import to be stronger when one considers scientific theories at some ideal final stage of development. Cf. footnote 52 for some remarks on this point.

[19] This makes the meaning of 'epistemic' in 'epistemic indispensability argument' broader than that we ascribed to it in [Panza and Sereni, 2013], ch. 6, and in [Panza and Sereni, 2015]. This will become clear in § 3.6.

[20] It goes without saying that arguments in which 'P' is replaced by an ontological property are likely to presuppose some robust realist understanding of scientific theories as delivering faithful representations of some external reality in agreement with some experimental practice, while arguments in which 'P' is replaced by an ontological property may be compatible with weaker conceptions, where scientific theories are conceived, for instance, as giving reliable descriptions limited to the observable world and/or as allowing for sufficiently reliable predictions of empirical phenomena.

A natural candidate for a property featuring in ontological arguments is truth (a theory being taken to be true if all its consequences are). Other candidates are approximate truth, truthlikeness, high objective probability, and the like. A natural candidate for a property featuring in epistemic arguments is being rationally justified. Other candidates are (empirical) confirmation, corroboration, high subjective probability, and the like.

In order to determine 'P' fully it will not be enough, however, just to choose one of these properties, since all of them can be, and are, conceived of in different ways: switching from one conception to another can have dramatic consequences for the soundness of genuine IA's. For example, taking some scientific theories to be true on the basis of a disquotational notion of truth has certainly not the same consequences than taking them to be true on the basis of a conception of truth as correspondence to an external reality. Again, taking the justification of some scientific theories to be justification of their truth is different from taking it to be justification of their successfulness or appropriateness for some given purposes. As a consequence, when one is to pass from $\mathsf{Sc.IA}_1^O$, $\mathsf{Sc.IA}_1^E$, $\mathsf{Sc.SIA}_1^O$ or $\mathsf{Sc.SIA}_1^E$, to genuine or strengthened IA's, one must supply a clarification of the conception embraced.

One could argue that this difficulty could be avoided by eliminating the parameter 'P' from $\mathsf{Sc.IA}_0$ and $\mathsf{Sc.SIA}_0$, and taking the relevant scientific theories to be just "our best scientific theories", as it happens in many cases (e.g. both in the "scientific" and the "Quine/Putnam" indispensability arguments offered by Colyvan, and in Baker's "enhanced" one: [Colyvan, 2001], pp. 7 and 11; [Baker, 2009], p. 613). If the final purpose is to get sound genuine IA's, this move seems inadequate if it is not also clarified which theories these are or what makes them our best ones; and if one just wanted to provide a list of these theories, one should also explain what lets them verify the appropriate instances of premises (*ii*) and (*iii*) of $\mathsf{Sc.IA}_0$ or of $\mathsf{Sc.SIA}_0$, and this could hardly be done without ascribing to them a property like those that 'P' could be intended to designate. The same holds if one wishes to spell out the reasons why these theories are our best ones. For in order to obtain a sound genuine IA, one must thereby show that the relevant theories verify the appropriate instances of these premises, and this can be so, again, only if those reasons rely themselves on the fact that these theories have such properties. In both cases, then, one is led back to the problem of specifying a property that 'P' is intended to designate: the mere claim that the relevant theories are our best ones cannot but be a shortcut for saying that they have such a property[21].

[21] Assessment of the virtues of scientific theories is commonly made through an array of properties, such as simplicity, ontological parsimony, unificatory power, familiarity of principles, fruitfulness, and so on. It is then plausible to consider a scientific theory to be one of our best insofar as it possesses some or all of these virtues to some specified degree. Still, the problem in the present context is not just on which of these grounds a scientific theory is taken to be among our best. Rather, the problem is to consider whether its being among the best entails that it has, or depends on its having, a certain property. A crude example will clarify the matter here. Suppose we suggest to enlist a scientific theory among our best merely because of its extreme simplicity. However we evaluate this choice, it should be clear that this will not be enough for a mathematical theory indispensable to it to be true,

3.4 Indispensability of What?

Section summary. We discuss possible replacements for parameter 'Q' and surveys alternative possibilities of restricting attention to mathematical Q's according to different conception of what mathematics could be taken to be in the context of an IA. We also explore which features of mathematics may be said to be indispensable to scientific theories in IA, and argue that different options may be reduced to just one option, that of replacing 'mathematical Q's' by 'mathematical theories'.

The replacement clauses relative to the remaining parameters involved in $\mathbf{Sc.IA}_0$ directly depend on the meaning to be ascribed to 'mathematical', and 'indispensable' in premise (*ii*).

Let us begin with 'mathematical'. While it is obvious that speaking of mathematical Q's allows one to restrict the scope of the arguments, it is by far less obvious how the restriction is to be conceived. Is mathematics here to be understood as a fixed sphere of (abstract) thoughts, or as a system of eternal truths, or rather as a human activity taking place in history, or again as a mutable system of results actually attained? Suppose we take it as a fixed sphere of (abstract) thoughts, or as a system of eternal truths. This may end up making some versions of IA circular. This is would be the case for those versions whose conclusion were that some mathematical theories, or some mathematical statements are true. For in order to get this conclusion, one should take mathematical Q's to be mathematical theories, or mathematical statements, and the relevant instances of premise (*ii*) of $\mathbf{Sc.IA}_0$ or $\mathbf{Sc.SIA}_0$ could, then, be argued for only by granting that these theories or statements are true (indeed, if this is not granted, under this understanding of 'mathematical' they could not be taken to be mathematical). Moreover, adopting this understanding of mathematics would confine the purpose of any other version of IA to that of supporting one particular picture of the relevant sphere of thoughts or system of truths, as opposed to others (for instance, a picture of the truths of mathematics as eternal truths about eternal objects, as opposed to purely logical truths lacking any objectual content). Though this last outcome may be seen as relevant enough, the (original) spirit of IA suggests that it is not what is expected. It thus seems more judicious to follow the second direction and view mathematics as a human activity or a mutable system of actual results.

The question is then what feature of this activity or system is to be considered in an IA. According to premise (*ii*) of $\mathbf{Sc.IA}_0$, such a feature of mathematics will have to be somehow indispensable to the relevant scientific theories. If this is so, then according to premise (*iii*) of $\mathbf{Sc.IA}_0$, if these scientific theories have some relevant property P, then some sort of items involved in this feature of mathematics meet some (philosophically) significant condition. By determining 'Q', we fix the relevant feature of mathematics (conceived as above); by determining '$\mathfrak{L}$', we fix

or for the objects this latter theory is putatively about to exist. Something more than mere simplicity should be required for this to obtain.

the relevant modality of indispensability; by determining 'a', we fix the sort of items at issue; finally, by determining '$\mathcal{A}$', we fix the condition to be met by these items. It is then clear that, for a genuine IA to be sound, or even for its premises to be at least plausible, the determinations of all these parameters must all fit together appropriately.

There are several features of mathematics that can be taken into account, many of which are actually employed in most common formulations: the quantification over mathematical "entities" or (putative) "objects" (like in [Putnam, 1971]); mathematical entities or objects themselves (like in [Colyvan, 2001], p. 11, or [Baker, 2005], or many others); the apparent reference to such entities or objects (like in [Colyvan, 2001], p. 7); the assumption of the truth of some mathematical statements (like in Resnik "pragmatic" IA: [Resnik, 1995], pp. 169-171; [Resnik, 1997], pp. 46-48)[22]; mathematical vocabulary (which we take to be what is often implicitly intended when authors use 'apparent reference')[23]; or finally, mathematical theories.

These options result from replacing in $\mathbf{Sc.IA}_0$ 'mathematical Q's' respectively with: 'mathematical theories'; 'mathematical quantifications' (used to designate occurrences of quantifiers whose range includes mathematical entities or objects); 'mathematical entities' or 'mathematical (putative) objects'; 'mathematical terms' (used to designate singular terms which are supposed to refer); 'mathematical constants' (used to designate constants belonging to a mathematical vocabulary), 'mathematical putative truths' (used to designate some mathematical statements held to be true).

Some may acknowledge significant differences among these six options, and argue that they may affect the arguments to follow. Against this, in [Panza and Sereni, 2013], pp. 204-205, we have argued that these are nothing but terminological variants of replacing 'mathematical Q's' with 'mathematical theories'. A detailed survey of the various options at stake may involve some lengthy considerations. For the sake of simplicity, we merely assume here that all these options can be reduced to this latter. In order to support this choice, beside relying on our discussion in that work, let us observe that there seems to be a common idea underlying these options, i.e. that

[22] Resnik's argument is hard to accommodate not only with $\mathbf{Sc.IA}_0$ but also with any valid argument schema. By eliminating a number of logically unessential ingredients, we take it to be as follows:

[Resnik's pragmatic argument]

i) Some scientific theories are justifiably used (by us).

ii) Among them, some are such that some mathematical putative truths are indispensable to them.

iii) If some putative truths are indispensable to some scientific theories which are justifiably used (by us), then these putative truths are to be justifiably taken to be actually true (by us).

Hence

iv) The mathematical putative truths mentioned in (*ii*) are to be justifiably taken to be actually true (by us).

We discussed Resnik's argument in some more details in [Panza and Sereni, 2015].

[23] This seems to be the case, for instance, in [Colyvan, 2001], p. 16

the relevant scientific theories make essential recourse to a vocabulary fixed by some mathematical theories, and then to the notions that this vocabulary is supposed to convey. Abstracting from mathematical theories, by focusing either on mathematical entities, mathematical language, or mathematical statements as such, seems to us to raise the danger of making the argument circular. For no entity can, in itself, be taken to be indispensable to something in whatever way, if it is not taken to exist (at least according to the usual conception of existence as a primitive and irreducible condition that is admitted here, and to the usual supposition that nothing can be so and so if it does not exist in this sense); and no language or statement can be taken to be meaningful by itself (that is, independently of some stipulations whose admission would just consist in adopting a theory having this language as its language) if it is not taken to be able to speak of something we can have an epistemic access to independently of this very language.

In our opinion, there is then no other appropriate option than basing an IA on the acknowledgment that some mathematical theories (or better, the recourse to them) are (is) somehow indispensable to some scientific theories. Hence, 'Q' is indeterminate only with respect to how some scientific theories are to be taken to have recourse to some (other) theories. Is it enough, for example, to borrow part of the vocabulary of these letter theories, or should we rather see these theories as entirely included within the relevant scientific ones? And, in the former case, is it enough to borrow some constants included in this language, or should we also require that the intended semantic of this language, or a relevant part of it, are adopted?

3.5 What is Indispensability?

Section summary. Various possible replacements for parameter '$\mathfrak{L}$' are discussed, in order to throw light on the notion of indispensability. We first consider the case where a single mathematical theory is said to be indispensable to another single scientific theory. Several examples are offered, and Baker's Enhanced Indispensability Argument is discussed. The notion of indispensability is found wanting of various specifications. We suppose that any theory (either scientific, or mathematical, or of any other sort) can have different instances (or admit different formulations), and we argue that a certain theory (either mathematical or not) can appropriately be said to be indispensable for a certain scientific theory only insofar as any instance of the latter must indispensably have recourse to an instance of the former in order to accomplish a certain task in an appropriate way. By supposing that a scientific theory can be said to accomplish a certain (descriptive, predictive, explanatory, etc.) task if it meets some condition h, and to do that in an appropriate way if it also meets some condition k, and that accomplishing this task in such an appropriate way is a necessary condition for this scientific theory to have the property P ascribed to it in the premise (i) of an IA, we suggest a first definition of indispensability in terms of h and k. Having properly introduced the notion of a family of theories, generalizations of this definition are then considered for application to cases where indispensability relates families of mathematical theories to families of scientific theories, single mathematical theories to families of scientific theories, and families of mathematical theories to single scientific theories.

Let us come now to the replacement clauses for '$\mathfrak{L}$'. Most of IA's currently discussed do not address the need of specifying any particular modality under which the relevant mathematical Q's are said to be indispensable to scientific theories (with the relevant exception of Baker's Enhanced Indispensability Argument, to which we will come soon[24]): they just state that the former are indispensable to the latter. This is quite strange, however, since, when one looks carefully at it, the very idea of the indispensability of something for scientific theories (conceived as said in § 3.3) is far from clear. Let us try then to explore the matter further.

Up to now, we have spoken of the indispensability of some Q's (crucially some mathematical Q's) for some scientific theories, both in the plural. Odd as this may have seemed, it depended on our intent to remain open both to the possibility of taking these Q's and scientific theories singularly—that is, each of them separately from other Q's and scientific theories, respectively—, or altogether, as forming appropriate families of Q's and scientific theories, respectively.

In order to make this point clearer, let us confine ourselves to the case where the Q's are theories (even though what follows could apply, *mutatis mutandis*, to other options, if considered to be significantly different)[25]. The question is then whether IA's are to be concerned with the indispensability of single mathematical theories for single scientific theories, or with the indispensability of families of mathematical theories for families of scientific theories, or even with the indispensability of single mathematical theories for families of scientific theories, or of families of mathematical theories for single scientific theories[26].

[24]Before Baker's argument, Colyvan's discussion of "the role of confirmation theory" (cf. [Colyvan, 2001], pp. 78-81) displayed awareness that the notion of indispensability may be somehow relational in character. We take our following discussion to improve on that suggestion. It still remains, however, that whatever relation character indispensability may be thought to have by Colyvan, it was not displayed in the very formulation of arguments he discusses, contrary to what happens in Baker's argument and, more explicitly, in our versions of IA.

[25]Notice that we leave open how recourse of a scientific theory to a mathematical theory is to be understood.

[26]In order to avoid confusion, let us make clear from the beginning that we do not take a family of theories as the mere conjunction of the elements that these theories are formed by, but rather as a system of separate (though possibly related) theories, each of which is considered as self-standing. If we suppose that theories are nothing but bodies of statements, the former option would consist in taking families of theories to be in turn theories. The latter option consists in taking them to be systems of bodies of statements, each of which remains distinct from each other, even when the family is considered as a whole. Clearly, the latter is the only plausible option if we want to consider families of theories composed by rival, or even incompatible theories. Moreover, it is only under this option that it makes sense to distinguish between the case in which IA concerns theories from that in which it rather concerns families of theories.

The first option is arguably the most natural one, but the others are suggested by the generic appeal, in many current IA's, to the indispensability of the quantification over mathematical entities, or of mathematical entities *tout court*, or by the use of expressions like 'indispensable to science' (and also 'indispensable to our best scientific theories').

Let us begin by considering this first option[27]. According to it, in $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$, 'some scientific theories' and (appropriate determinations of) 'some (mathematical) Q's' will respectively refer to a number of scientific theories and to a number of (mathematical) theories, and each of the former is considered in virtue of its having recourse to just one of the latter. Consequently, in $^*\mathbf{Sc.IA}_0$ and $^*\mathbf{Sc.SIA}_0$, 'ScTh' and 'Q' will stand respectively for the properties of being a single scientific theory and a single (mathematical) theory.

Saying that a certain theory (mathematical or not) is indispensable for a certain scientific theory seems to us to be just a *façon de parler* for saying that the latter must indispensably have recourse to the former if it has to accomplish a certain task in an appropriate way[28]. Asserting that a theory T is indispensable to a scientific theory S cannot but mean, therefore, that if T were missing, or were to be replaced by any other theory T^*, this would result in either maintaining S unchanged but making it unable to accomplish its relevant task in the appropriate way, or in transforming S in a different theory S^* (that could either be able to accomplish its task in the appropriate way or not). Moreover, in the context of an IA, accomplishing this task in an appropriate way has to be considered as a necessary condition for this scientific theory to have the property P ascribed to it in premise (i). As a consequence, if we want to determine what is meant exactly when we speak of indispensability in the context of IA, we need to get clear on: i) the identity conditions of the two theories; ii) the task that the scientific theory is supposed to accomplish; iii) the appropriate way in which it is required to accomplish such task; iv) and, therefore, the property P that it is supposed to have.

The point is not undermined even by taking T to be indispensable to the mere articulation or formulation of S. For also in this case it it is hard to see what this could mean in the end, if not that if T were missing or were to be replaced with any other theory T^*, S would be transformed in another theory S^*, or it would be made unable to accomplish some relevant task in the appropriate way.

A well-known example of an IA where at least one of these four aspects is explicitly taken into account (namely the task that the relevant scientific theories are supposed to accomplish) is Baker's "enhanced" argument ([Baker, 2009], p. 613; Baker was partly anticipated in this by [Field, 1989], pp. 15)[29]:

[27] This is the only option we have taken into account in [Panza and Sereni, 2013].

[28] Cf. footnote 52, below, for the sense in which we are talking of task here.

[29] Baker's argument is meant to be enanched with respect to the argument put forward by [Colyvan, 2001], p. 11:

[**Baker's Enhanced Indispensability Argument**]

i) We ought rationally to believe in the existence of any entity that plays an indispensable explanatory role in our best scientific theories.

ii) Mathematical objects play an indispensable explanatory role in science.

Hence

iii) We ought rationally to believe in the existence of mathematical objects.

We take premise (*ii*) to mean that our best scientific theories would not be able to be as explanatory as they are supposed to be, if they did not involve mathematical objects is some way. It still remains, however, that in order to make this claim completely clear, one should not only explain what it means that a scientific theory is as explanatory as it is supposed to be, but also: how it is required to be in order to accomplish this task in an appropriate way; which property it is supposed to have (for which 'P' stands in $\mathsf{\textbf{Sc.IA}}_0$ and $\mathsf{\textbf{Sc.SIA}}_0$) in order for a sound genuine IA to apply to it; and what it means that it would not remain the same if it did not somehow involve any mathematical objects.

Baker's discussion of his argument and his famous examples from the life cycle of periodical cicada (*ibid*, p. 614) possibly tell us something on the first question. The remaining three questions, however, seem not to be sufficiently explored. Moreover, Baker's argument seems concerned, at least on its face, with the indispensability of mathematical objects in general, and this, at least on one plausible reading, suggests that he is thinking of the indispensable explanatory role of the whole family of accepted mathematical theories (granted, of course, that he is not circularly thinking of the indispensability of mathematical objects considered by themselves, independently of any theory)[30]. Far from fully determining the notion of indispensability, Baker's argument urges then us to shed light on the different aspects of this notion that stand in need of further determination.

Suppose we agree that both a certain scientific theory S (conceived as specified in § 3.3) and a certain other theory T can admit of a number of distinct formulations or ways of presentations—which we shall respectively call 'instances of S' and 'instances of T'[31]. Suppose also that we have fixed a way in which an instance of a scientific theory can possibly have recourse to an instance

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[**Colyvan's Indispesability argument**]

i) We ought to have ontological commitment to all and only the entities that are indispensable to our best scientific theories.

ii) Mathematical entities are indispensable to our best scientific theories.

Hence

iv) We ought to have ontological commitment to mathematical entities.

Baker's argument is meant to enhance Colyvan's argument exactly in so far as it makes it explicit that there is a specific task that scientific theories, and mathematical theories (or, allegedly, objects) indispensable to them, are meant to accomplish, and that this task is an explanatory one.

[30]Concerning the circularity of this attitude, cf. section 3.4.

[31]It is assumed here that we have an ability of recognising canonical instances of the relevant theories. This is

of another theory. Finally, suppose that an instance of S is taken to accomplish a certain task if it meets a certain condition h and that it is taken to do it in an appropriate way if it also meets a certain condition k. We shall say that:

T is h-k-dispensable to S if and only if there is an instance S_i of S such that S_i meets both the conditions h and k and has no recourse to any instance of T in the way fixed.

T is h-k-indispensable to S if and only if it is not h-k-dispensable, that is, if and only if for any instance S_i of S which meets both the conditions h and k, there is an instance T_i of T such that S_i has recourse to it in the way fixed.

Imagine, for example, that S has an instance S_0, with a certain empirical content, that is deductively well-balanced—that is, it admits a reasonably small numbers of assumptions stated in a reasonable simple language from which all its consequences deductively follow, possibly on the basis of the ascertainment of certain empirical data—, and has recourse to an instance of T in the way that has been fixed. Let h be the condition of having the same empirical content as S_0, or possibly a wider one, and k be the condition of being deductively well-balanced. Then T is h-k-dispensable to S if and only if there is an instance S_i of S, such that S_i has the same empirical content as S_0, or possibly a wider one, is deductively well-balanced, and has no recourse to any instance of T in the way that has been fixed. T is thus h-k-indispensable to S if and only if for any instance S_i of S which has the same empirical content as S_0, or possibly a wider one, and is deductively well-balanced, there is an instance T_i of T such that S_i has recourse to it in the way that has been fixed.

An immediate consequence of this definition is that T is, as it were, vacuously h-k-indispensable to any theory S no instance of which meets both the conditions h and k, for in this case the implication 'S_i meets the conditions h and $k \Rightarrow$ there is an instance T_i of T such that S_i has recourse to it in the way that has been fixed' is vacuously true since its antecedent is always false. One could try to avoid this unpleasant consequence of the definition by amending it in some appropriate way, for example by restricting its range to scientific theories admitting at least an instance that meets both the conditions h and k. Nonetheless, if the aim of a clarification of the notion of indispensability is to discuss the soundness conditions of genuine IA's, this apparently obnoxious consequence has no undesirable effects: we have required above that the conditions h and k and the property P have to be so determined that an instance of a scientific theory S meets both these conditions only if S is P. It then follows that a theory S no instance of which meets both conditions is not P and cannot then be involved in a sound genuine IA.

This definition takes the identity conditions of the relevant theories as given in advance, as well as the conditions for two instances of a theory to count as instances of the same theory. A more general definition would go as follows.

indeed required if we are to be able to identify these theories in the scientific and mathematical practice in which they should be immersed, in accordance with what have been said in §§ 3.3 and 3.4.

Suppose that S_0 and T_0 are two instances of S and T, respectively, and admit that S'_j and T'_j are instances of S and T if and only if S'_j is equivalent to S_0 according to an equivalence relation σ and T'_j is equivalent to T_0 according to an equivalence relation τ. Then:

T is h-k-dispensable$_{\sigma\text{-}\tau}$ to S if and only if there is an instance S'_i of a scientific theory that is σ-equivalent to S_0, meets conditions h and k, and has no recourse to any instance of a theory which is τ-equivalent to T_0; T is otherwise h-k-indispensable$_{\sigma\text{-}\tau}$ to S.

If we suppose that the way in which a scientific theory has recourse to an instance of another theory has been determined via a full determination of 'Q', then—in the case where $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$ concern the indispensability of single theories for single scientific theories—the full determination of the parameter '$\mathfrak{L}$' is obtained via a determination of conditions h and k, in agreement with the determination of the property P, and of the two equivalence relations σ and τ (we are considering here the case in which $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$ concern the indispensability of single theories to single scientific theories).

Assume then that the way in which an instance of a scientific theory can have recourse to an instance of another theory and the two equivalence relations σ, τ have appropriately been fixed, and that both the relevant task on which h depends, and the appropriate way of accomplishing this task on which k depends, have been so determined that a scientific theory cannot accomplish this task in this way if it is not P. This allows one to speak without further qualification of the h-k-indispensability of a theory T for a scientific theory S, and of both these theories as being the same as, or distinct from, some theories $\mathsf{T}^\star$ and $\mathsf{S}^\star$.

The two following cases can now be considered.

First, suppose that a certain theory T is h-k-indispensable to a certain scientific theory S, but that there is a distinct scientific theory $\mathsf{S}^\star$ which accomplishes the same relevant task as S in the appropriate way—that is, as well as S does, or even better—, and which is such that T is h-k-dispensable to it. This means that there is an instance $\mathsf{S}^\star_i$ of $\mathsf{S}^\star$ that has recourse to no instance of T in the fixed way and meets conditions h and k. Suppose also that S satisfies premise (i) of $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$ under an appropriate determination of 'P'. This entails that $\mathsf{S}^\star$ is P too. Should then one say that T is h-k-indispensable to a family of scientific theories including both S and $\mathsf{S}^\star$? Our intuition is that one should not[32].

This suggests a new definition:

[32]The former option mentioned in footnote (26) would probably support a different intuition. But under this option, T would be h-k-indispensable to a family of scientific theories including both S and $\mathsf{S}^\star$ just for its being h-k-indispensable to S, so that there would be no point in considering $\mathsf{S}^\star$. Moreover, in the case considered here it should be possible that S and $\mathsf{S}^\star$ be rival theories; this is compatible with considering a family including both theories only under the first option of footnote (26). Taking T as h-k-dispensable to a family of scientific theories including both S and $\mathsf{S}^\star$ expresses thus the idea that having recourse to T is not indispensable for accomplishing the relevant task in the appropriate way; and this is, of course, all that is relevant in the case considered.

T is h-k-dispensable to a family $\mathscr{S}$ of scientific theories accomplishing the same relevant task, if $\mathscr{S}$ includes a scientific theory to which T is h-k-dispensable (even if it also contains other scientific theories to which it is h-k-indispensable); T is h-k-indispensable to $\mathscr{S}$ if it is not h-k-dispensable.

By way of example, suppose that, *pace* Field, real analysis were indispensable to Newton's gravitational theory, this latter being regarded as a well-balanced theory making exact predictions (with an acceptable margin of error) concerning astronomic (and possibly other non-microscopic) phenomena. One could still argue that the same exact predictions (with the same margin of error) concerning the same phenomena can also be made by another well-balanced theory which, by involving no appeal to irrational real numbers (but only, say, to rational ones), would be essentially distinct from Newton's gravitational theory. Our intuition here is that one should say that real analysis is dispensable to well-balanced scientific theories whose aim is to make exact predictions concerning these phenomena[33]

Second, suppose that a certain theory T is h-k-dispensable to a scientific theory S, so that there is an instance S_i of S that has no recourse to any instance of T in the fixed way, and meets conditions h and k. It is still possible that S_i has recourse to an instance of another theory $\mathsf{T}^\star$ of the same sort of T (for example to another mathematical theory, if T is mathematical), and that any instance of S which meets these conditions has either recourse to an instance of T or to an instance of $\mathsf{T}^\star$. Should then one say that a family of theories of the same sort composed by T and $\mathsf{T}^\star$ is h-k-dispensable to S? Again, our intuition is that one should not.

This suggests a new definition:

[33] The example is well suited for emphasizing the importance of relativizing indispensability as we are suggesting. For it would be natural to argue that the pervasive use of differential and integral calculus (especially differential and partial differential equations) in so many scientific theories (not only physical ones) makes real analysis (or whatever appropriate theory of mathematical continuum) indispensable for these very theories. The point would be certainly well taken, since no scientific theory akin to classical mechanics, in its usual formulation coming from Newton, Lagrange, and Hamilton, could, for example, avoid appealing to the differential and integral calculus without losing its very identity. Still, this form of indispensability is far from supporting appropriate versions of IA, since appealing to it in order to draw conclusions about mathematics would reduce to arguing that mathematics is as these conclusions claim it to be because it happened that some scientific theories are just as they happen to be, which is certainly not what IA is intended to conclude (though one could imagine other essentially different arguments along this line). Notice, incidentally, that, as convincingly maintained in [Maddy, 1997], ch. II.6, the use of differential and integral calculus in science goes typically together with the appeal to different sorts of "idealizations" (cf. also [Maddy, 1992] on idealizations, and [Colyvan, 2001, ch. 5] and [Leng, 2010, ch. 5] for discussions). And this is enough to undermine the possibility of taking the "indispensable appearance of an entity in our best scientific theory to warrant the ontological conclusion that it is real", since "for this conclusion, the appearance must be in a hypothesis that is not legitimately judged a 'useful fiction', in other words, in one that has been 'experimentally verified' [...], and it must be in the context that is not an explicit idealization" (*ibid.*, p. 152).

A family of theories $\mathscr{T}$ of a certain sort is h-k-indispensable to a scientific theory S if any instance of S which meets both h and k: (i) has recourse to an instance of either one or another among the theories in $\mathscr{T}$, (ii) for every theory T in $\mathscr{T}$ there is an instance of S that has recourse to at least an instance of T; $\mathscr{T}$ is of course h-k-dispensable to S if it is not h-k-indispensable to it.

To borrow an example from Molinini's contribution to this volume (cf. [Molinini, 2015]), special relativity may admit both a well-balanced instance offering good explanations of the Lorentz-contractions having recourse to ZF set-theory considered on its own, and a well-balanced instance offering the same good explanation having recourse to Minkowsky's geometry (taken as a primitive mathematical theory, unreduced to ZF); and it may be the case that all of its instances has recourse to either of these two theories and to no other mathematical theory. According to our definitions, a family of mathematical theories composed by ZF (considered on its own), and Minkowsky geometry (taken as primitive), is indispensable to special relativity regarded as a well-balanced theory able to provide a good explanation of the Lorentz-contractions (although neither ZF not Minkowsky geometry, individually taken, are so).

These two latter definitions finally suggest another definition:

A family $\mathscr{T}$ of theories of a certain sort is h-k-indispensable to a family $\mathscr{S}$ of scientific theories if for any theory S in $\mathscr{S}$, there is a subfamily of $\mathscr{T}$ that is indispensable to S, and any theory of $\mathscr{T}$ is included in a subfamily of it that is indispensable to a theory of $\mathscr{S}$; $\mathscr{T}$ is h-k-dispensable from $\mathscr{S}$ if it is not h-k-indispensable to $\mathscr{S}$.

The previous definitions, together with what we said about the replacement clauses for parameters 'P' and 'Q', should be enough for clarifying what has to be determined in order for premises (i) and (ii) of $\mathbf{Sc.IA}_0$, and $\mathbf{Sc.SIA}_0$ to be transformed in genuine non-schematic statements with a definite truth-value. One thing to do is to decide whether the indispensability relation involved in premise (ii) relates single theories to single theories or families of theories to families of theories, or even a single theory to a family of theories or a family of theories to a single theory. This has, of course, a consequence on the exact meaning to be ascribed to the expression 'some scientific theories' and 'some Q's' in both argument schemas[34], and then on the meaning of the corresponding existential assumptions. In other words, one has to decide whether 'some scientific theories' and 'some (mathematical) Q's' in $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$ are to refer respectively to a number of scientific theories and to a number of (mathematical) theories each of which is individually considered, or else to a family of scientific theories and to a family of (mathematical) theories[35].

[34] And thus also on the exact meaning to be ascribed to the predicate 'ScTh' and 'Q' in $^*\mathbf{Sc.IA}_0$ and $^*\mathbf{Sc.SIA}_0$.

[35] And thus whether 'ScTh' and 'Q' in $^*\mathbf{Sc.IA}_0$ and $^*\mathbf{Sc.SIA}_0$ are to stand respectively for the properties of being a single scientific theory and a single (mathematical) theory, or for the properties of being a family of scientific theory and a family of (mathematical) theories.

Making a choice or another can have dramatic consequences on the truth conditions of instances of premises (*ii*) and (*iii*)[36]. For example, if we assume that the indispensability relation relates either a single theory or a family of theories to a family of scientific theories, and the latter family is taken to be composed by the totality of scientific theories widely accepted within the scientific community—a quite natural understanding of the common expression 'our best scientific theories'—, it is all but clear that there will be true instances of premise (*ii*) (however the determination of '$\mathfrak{L}$', the way a scientific theory is taken to possibly have recourse to a mathematical one, and the relevant family of mathematical theories are determined). Analogously, if we assume that the indispensability relation relates a family of theories either to a single scientific theory or to a family of scientific theories, and the former family is taken to be composed by the totality of mathematical theories widely accepted within the mathematical community, it is clear that no instance of this premise will be true. *A fortiori*, this is also the case if the indispensability relation is taken to relate a family of theories to a family of scientific theories, and the former is taken to be composed by the totality of mathematical theories widely accepted within the mathematical community and the latter by the totality of scientific theories widely accepted within the scientific community. In other terms, any instance of premise (*ii*) seems to be untrue if it is respectively taken to assert that mathematics as a whole is $\mathfrak{L}$-indispensable to science as a whole, or that mathematics as a whole is $\mathfrak{L}$-indispensable to a part of science; and it is all but clear that there will be true instances of premise (*ii*) if it is taken to assert that part of mathematics is $\mathfrak{L}$-indispensable to science as a whole[37].

On the other hand, it will become clear in § 3.6 that if we take the indispensability relation to relate a family of theories either to a single scientific theory or to a family of scientific theories, it becomes quite hard to get instances of premise (*iii*) which are true[38].

3.6 IA's for Platonism, Veridicalism, and Externality

Section summary. Suitable replacements for parameters 'a' and '$\mathcal{A}$' are discussed. Main examples of replacements for 'a' are '(putative) objects' and 'theorems'. Re-

[36]For premise (*i*), things seem to be simpler, since it is reasonable to admit that a family of scientific theories is P if and only if any theory of this family is individually P (the underlying thought being that those properties for which this would not be the case would not be allowed for as possible replacements for 'P').

[37]Also in the cases just considered the first option mentioned in footnote (26) would probably suggest a different intuition. But, then, a family of theories would be $\mathfrak{L}$-indispensable to a family of scientific theories, or a theory to a family of scientific theories, or a family of theories to a scientific theory, just in case this were the case for one theory of the former family and one theory of the latter. Hence, for example, mathematics as a whole would be $\mathfrak{L}$-indispensable to a science as a whole, just in case that a mathematical theory were so for a scientific theory; but then it would somehow be misleading to speak of science and mathematics as wholes, rather than speaking of specific theories

[38]See, however, remarks at pages 38-39, below.

placements of '$\mathcal{A}$' are then discussed for each of these replacements of 'a' in both ontological and epistemic schematic IA, according to possible replacements of the already partially specified parameters 'P^O' and 'P^E' (cf. Section 3.3). Four argument schemas are singled out through further specifications: ontological IA's for platonism; epistemic IA's for platonism; ontological IA's for veridicalism; epistemic IA's for externality. Externality is here understood as the thesis that the relevant statements or theorems are externally justified, and external justification, as opposed to internal justification, is understood as a justification for statements that is not exhausted by consequences of a given theory (internal and external truth are also clarified along similar lines). Finally, two other argument schemas are acknowledged as 'mixed IA for platonism' and 'mixed IA for externality'. It is also observed that each of the foregoing six argument schemas can be given for strengthened IA's too.

The last step in getting a genuine IA is the full determination of the parameters 'a' and '$\mathcal{A}$'.

Let us begin with the former. At the best of our knowledge, all available versions of IA result from taking 'a' to be replaced either with 'entities' or 'objects', clearly understood as denoting putative objects—as it happens in all examples mentioned up to now, except for Resnik's "pragmatic" IA—, or with 'theorems', 'consequences', 'statements', or 'truths', clearly understood as denoting putative truths—as it happens in Resnik's argument[39].

Hence, if one takes Q's to be theories, as we suggested in § 3.4, their a's will respectively be either the entities or objects which these theories are putatively about or they putatively fix, or as the consequences (i.e. the theorems) of these theories.

If one follows instead one of the other options mentioned in § 3.4—thus suggesting significant differences with the one we adopted—, things will have to be specified case by case[40].

Once the replacements clauses for 'a' have been fixed, those for '$\mathcal{A}$' depend on them and on whether '$\mathcal{A}$' occurs either in an ontological schematic IA or in an epistemic schematic IA's. Four different options arise: '$\mathcal{A}$' is to be determined (1) in an ontological schematic IA in which 'a'

[39] Cf. footnote 22, above.

[40] If 'a' is replaced with 'entities' or '(putative) objects', then: the a's of the relevant quantifications will be the entities or putative objects that the corresponding quantifiers range on; the a's of the relevant entities or putative objects will be these very entities or putative objects; the a's of the relevant terms will be the entities or putative objects that these terms putatively refer to; the a's of the relevant constants will be the entities or putative objects that these constants are used to speak of (either by putatively referring to them, or by designating some properties of, or relation among them, or some functions defined on them); finally, the a's of the relevant putative truths will be the entities or putative objects of which these putative truths are supposed to be true. If 'a' is replaced with 'statements' or 'consequences', then: the a's of the relevant quantifications will be the corresponding quantified statements; the a's of the relevant entities or putative putative objects will be the relevant statements concerning these entities or putative objects; the a's of the relevant terms will be the relevant statements involving these terms; the a's of the relevant constants will be the relevant statements involving these constants; finally, the a's of the relevant putative truths will be these very putative truths.

is replaced with 'entities' or '(putative) objects'; (2) in an ontological schematic IA in which 'a' is replaced with 'statements' or 'consequences'; (3) in an epistemic schematic IA in which 'a' is replaced with 'entities' or '(putative) objects'; (4) in an epistemic schematic IA in which 'a' is replaced with 'statements' or 'consequences'. These four different sorts of schematic IA's can hope to deliver genuine sound IA's only if '$\mathcal{A}$' is determined in each of them in an appropriate way.

In case (1) '$\mathcal{A}$' has to stand for a condition that appropriate entities or objects putatively meet independently of our epistemic relation with the scientific theories which indispensably appeal to them in same way (if any). In case (2),'$\mathcal{A}$' has to stand for a condition that the consequences of a theory, or some appropriated statements, putatively meet, again independently of such an epistemic relation. In case (3) '$\mathcal{A}$' has to stand for a condition that appropriate entities or objects putatively meet because of such an epistemic relation. Finally, in case (4) '$\mathcal{A}$' has to stand for a condition that the consequences of a theory, or some appropriate statements, again putatively meet because of such an epistemic relation.

Let us consider cases (2) and (4). In these cases the natural choice seems to take '$\mathcal{A}$' to stand for truth in ontological schematic IA's, or for rational justification in epistemic schematic IA's. Still, one should not take for granted that 'true' and 'justified' means the same when applied, respectively, to scientific theories and to statements that may be somehow indispensably involved in scientific theories (e.g. to consequences of other theories that may be indispensable to scientific theories). Once more, their meaning will be a matter of the determination of the other intermediate parameters involved in $\textbf{Sc.IA}_0$, and $\textbf{Sc.SIA}_0$.

In order to make this clear, let us then distinguish truth and justification as putative properties of scientific theories from ϑ-truth and ϑ-justification as putative properties of statements that may be somehow indispensably involved in scientific theories.

Notice that ϑ-justification of the consequence of some theories must in any case be something else than just the property of being appropriately obtained within the theory (e.g. proven in it, if the theory is a mathematical one): this would be tantamount to say that they are what they are, namely consequences of the relevant theories. Their being ϑ-justified should rather depend on our having some rational reasons to attribute some additional virtue to them (for example, reasons for taking them to be true, according to a conception of truth for which truth does not coincide merely with being appropriately derived within a theory). In order to help ourselves with some ready-to-use terminology, when the justification of a certain statement depends on attributing to it a virtue that is not exhausted by its being consequence of a given theory, we shall then say that its justification is external (and internal otherwise). Our present point can thus be expressed by saying that ϑ-justification is an external justification.[41].

[41]The adjectives 'internal' and 'external' are reminiscent of other well-known debates. They are reminiscent of the distinction between internalism and externalism about knowledge and justification in epistemology; and they are reminiscent of the distinction between internal and external ontological questions as advanced in [Carnap, 1950]. Although similiarities between our use of these expressions and their use in these other debates may be suggested,

Something similar has also to hold for ϑ-truth. This cannot reduce to anything like internal truth, i.e. truth so conceived that a statement is true just insofar as it is a consequence of a certain theory. Whatever it might be, ϑ-truth must agree with an external notion of truth, i.e. truth depending on some sort of correspondence with an independent reality.

In cases (2) and (4), it seems then quite natural that '$\mathcal{A}$' should stand for ϑ-truth and ϑ-justification, in ontological and epistemic IA's respectively, whatever property 'P^O' and 'P^E' might stand for. Suppose, for example, that 'P^O' and 'P^E' stand respectively for approximate truth, truth-likeness, high objective probability, or the like, and for (empirical) confirmation, corroboration, high subjective probability, or the like. Should one maintain that if some scientific theories have one of these properties, then the mathematical statements somehow indispensably involved in them (if any) inherit this same property? This would surely contrast with the widely accepted idea that mathematics is neither imprecise nor contingent. We should thus reject this option[42]. Nonetheless we can still maintain that truth or rational justification of some scientific theories is inherited by the mathematical statements somehow indispensably involved in these scientific theories (if any). What could at best be claimed here, is that the truth—or, *a fortiori*, the approximate truth, truth-likeness, high objective probability, or the like—of these scientific theories entail some sort of rational justification of these mathematical statements, rather than truth.

In cases (1) and (3), the natural choice seems rather to take '$\mathcal{A}$' to stand for existence, for ontological schematic IA's, and for justifiably ascribed existence (*i.e.* the condition that something meets if we are rationally justified in taking it to exist), for epistemic IA's. In both cases, for analogous reasons, existence is to be understood as external existence, in the sense suggested above[43]. Once more, this is independent of the property that 'P^O' and 'P^E' stand for. The reason is similar to that considered above for cases (2) and (4). When their existence is concerned, taking mathematical entities or putative objects to be susceptible of any property respectively distinct from, and lees robust than, existence itself or justifiably ascribed existence (e.g. approximate existence, existence-likeness, high objective probability of existence, and the like; and confirmed ascribed existence, corroborated ascribed existence, high subjective probability of existence, and the like) seems to contrast with a widely accepted conception of mathematics. And again, using forms of IA in order to ascribe one of these properties to mathematical entities or putative objects,

and may have driven us in choosing them, our terminology should merely be understood here as technical jargon with no other meaning than the one we have explained.

[42]This is in no way an argument against the possibility of endorsing IA in a different setting, where mathematics is rather conceived as imprecise and/or contingent as empirical sciences. We have no intention to argue against such an approach. We rather confine ourselves to considering more traditional and usual forms of IA, where '$\mathcal{A}$' may stand either for ϑ-truth or for ϑ-justification. We shall just take it that what we shall say in the rest of this paper about these forms of IA also applies, *mutatis mutandis*, to other forms, agreeing with this alternative setting.

[43]In what follows we shall take the verb 'to exist' and its cognates in their external sense, as appears to be customary in the debate on IA.

on the ground that some scientific theories somehow indispensably appeal to them, seems to contrast with the spirit of IA. As with cases (2) and (4) above, one could at most concede that the truth—or, *a fortiori*, the approximate truth, truth-likeness, high objective probability or the like—of the relevant scientific theories can at best entail the justifiably ascribed existence of these entities of putative objects, rather than their actual existence[44]

We take, then, the conclusions of IA's relative to cases (1) to (4) to be respectively that: (C_1) the relevant mathematical entities or putative objects exist; (C_2) the relevant mathematical entities or putative objects may be justifiably taken to exist, that is, we are justified in taking them to exist; (C_3) the relevant mathematical statements or theorems are ϑ-true (that is, they are true in an appropriate, external sense); (C_4) the relevant mathematical statements or theorems are ϑ-justified (that is, they are justified in an appropriate, external sense).

It is important to notice that arguing that ϑ-justification has to be an external justification is not the same as requiring that it be justification for external truth (that is, justification for us to believe that the relevant statements are true). This can be seen quite naturally as an external justification, but it is not the only sort of external justification conceivable. For instance, one could say that some statements are externally justified if they are consequences of a theory that is in turn itself justified[45]; or else, one can say that they are externally justified if they are useful for some external task (that is, some task not merely involved in the internal development of a theory of which they are consequences). This suggests that an epistemic IA in which 'a' is replaced with 'statements' or 'consequences' is not necessarily an epistemic argument for veridicalism (cf. page 4, above): its conclusion could be that the relevant mathematical statements are externally justified without this justification being a justification for their truth. This sort of epistemic IA should more generally be considered as an argument in favor of what we will here call 'epistemic externality', i.e. the thesis (not to be confused with 'externalism' as used in the epistemological debate) that those statements are externally justified.

We have then the following four sorts of IA's: ontological IA's for platonism; epistemic IA's for platonism; ontological IA's for veridicalism; epistemic IA's for externality (a particular case of which will be given by an epistemic IA for veridicalism). By making the respective replacements in $\mathbf{Sc.IA}_1^O$ and $\mathbf{Sc.IA}_1^E$, we get the following four arguments schemas, which respectively correspond to these four sorts of IA's:

$\mathbf{Sc.IA}_2^{O,P}$, obtained from $\mathbf{Sc.IA}_0$ by replacing 'P' with 'P^O', 'a' indifferently with 'entities' or with 'putative objects', and 'meet the condition $\mathcal{A}$' with 'exist'.

[44]Again, this is no argument against the possibility of endorsing IA in a different setting. Far for excluding this possibility, we merely confine ourselves to consider more traditional and usual forms of IA, where '$\mathcal{A}$' may stand either for existence, or for justifiably ascribed existence, and we take it that what we shall say about these forms of IA also applies, *mutatis mutandis*, to other forms, agreeing with this alternative setting.

[45]This will become clearer in § 4.1

$\mathsf{\textbf{Sc.IA}}_2^{E,P}$, obtained from $\mathsf{\textbf{Sc.IA}}_0$ by replacing 'P' with 'P^E', 'a' indifferently with 'entities' or with 'putative objects', and 'meet the condition $\mathcal{A}$' with 'are to be justifiably taken to exist'.

$\mathsf{\textbf{Sc.IA}}_2^{O,V}$, obtained from $\mathsf{\textbf{Sc.IA}}_0$ by replacing 'P' with 'P^O', 'a' with 'statements' or 'consequences', according to the determination of 'Q', and 'meet the condition $\mathcal{A}$' with 'are ϑ-true'.

$\mathsf{\textbf{Sc.IA}}_2^{E,Ext}$, obtained from $\mathsf{\textbf{Sc.IA}}_0$ by replacing 'P' with 'P^E', 'a' with 'statements' or 'consequences', according to the determination of 'Q', and 'meet the condition $\mathcal{A}$' with 'are ϑ-justified'.

For each of these arguments schemas, one can get, then, another argument schema, by making the same replacements in $\mathsf{\textbf{Sc.SIA}}_0$, $\mathsf{\textbf{Sc.SIA}}_0$, which respectively give the four arguments schemas $\mathsf{\textbf{Sc.SIA}}_2^{O,P}$, $\mathsf{\textbf{Sc.SIA}}_2^{E,P}$, $\mathsf{\textbf{Sc.SIA}}_2^{O,V}$ and $\mathsf{\textbf{Sc.SIA}}_2^{E,Ext}$.

The fact that from the truth of the relevant scientific theories what can at best follow is either some sort of justification of the relevant statements or consequences, or the justifiably ascribed existence of the relevant entities or putative objects, suggests then two other possible argument schemas, which we shall call, respectively, 'mixed IA's for platonism' and 'mixed IA's for externality', namely:

$\mathsf{\textbf{Sc.IA}}_2^{M,P}$, obtained from $\mathsf{\textbf{Sc.IA}}_0$ by replacing 'P' with 'P^O', 'a' indifferently with 'entities' or with 'putative objects', and 'meet the condition $\mathcal{A}$' with 'are to be justifiably taken to exist'.

$\mathsf{\textbf{Sc.IA}}_2^{M,Ext}$, obtained from $\mathsf{\textbf{Sc.IA}}_0$ by replacing 'P' with 'P^O', 'a' with 'statements' or 'consequences', according to the determination of 'Q', and 'meet the condition $\mathcal{A}$' with 'are ϑ-justified'.

Again, for each of these arguments schemas one can obtain another argument schema, by making the same replacements in $\mathsf{\textbf{Sc.SIA}}_0$, $\mathsf{\textbf{Sc.SIA}}_0$, which respectively deliver the two arguments schemas $\mathsf{\textbf{Sc.SIA}}_2^{M,P}$ and $\mathsf{\textbf{Sc.SIA}}_2^{M,Ext}$.

One gets then, in total, six schematic IA's from $\mathsf{\textbf{Sc.IA}}_0$ and six schematic strengthened IA's from $\mathsf{\textbf{Sc.SIA}}_0$, obtained by partially determining the parameters 'P', 'a' and '$\mathcal{A}$'. We say 'partially determining' also with respect to 'a' and '$\mathcal{A}$' since the replacements that have been suggested for them do not still succeed in fully determining them, except, perhaps, in the case where 'a' is replaced with 'consequences' and 'Q' with 'theories'. This should already be clear from what has been said about ϑ-truth and ϑ-justification. But also speaking of the putative objects of a certain theory, and, even worst, of their existence (even granted that this has to be understood externally), is, for example, far from being free from ambiguity. Hence, in order to pass from these twelve schematic arguments to genuine arguments, one has much more to do than fully determining the parameters 'P^O' or 'P^E' and 'Q' and '$\mathfrak{L}$'.

But there is more than this. One could argue that, though possibly true, the premises (*iii*) of the schematic arguments for platonism—namely of $\mathbf{Sc.IA}_2^{O,P}$, $\mathbf{Sc.IA}_2^{E,P}$, $\mathbf{Sc.IA}_2^{M,P}$ and $\mathbf{Sc.SIA}_2^{O,P}$, $\mathbf{Sc.SIA}_2^{E,P}$, $\mathbf{Sc.SIA}_2^{M,P}$—stand in need of a rationale: why is it the case that from the truth of the relevant scientific theories (or, *a fortiori*, from their approximate truth, truth-likeness, high objective probability or the like), or from their being rationally justified (or, *a fortiori*, from their confirmation, corroboration, high subjective probability or the like), together with the fact that some Q's involving some entities or putative objects are indispensable to these theories, the existence, or the justifiably ascribed existence of these entities or putative objects should respectively follow? And, even worst, why should the inverse implication hold too?

To answer this question, consider the case where Q's are theories. One could split these premises in two as follows:

[$\mathbf{Sc.IA}_2^{O/E/M,P}$] *iii.i*) If some theories are $\mathfrak{L}$-indispensable to some scientific theories which are P, then they are R[46].

[$\mathbf{Sc.IA}_2^{O/E/M,P}$] *iii.ii*) If some theory are R, then their putative objects meet the condition $\mathcal{A}^{ex}$[47].

or

[$\mathbf{Sc.SIA}_2^{O/E/M,P}$] *iii.i*) Some theories are $\mathfrak{L}$-indispensable to some scientific theories which are P if and only if they are R[48].

[$\mathbf{Sc.SIA}_2^{O/E/M,P}$] *iii.ii*) Some theories are R if and only if their putative objects meet the condition $\mathcal{A}^{ex}$[49],

where 'R' is to be replaced with 'ϑ-true' or 'ϑ-justified' (assuming of course that a theory is 'ϑ-true' or 'ϑ-justified' if and only if its consequences are) and 'meet the condition $\mathcal{A}^{ex}$' is to be replaced with 'exist' or 'are to be justifiably taken to exist' if 'P' is is replaced with 'P^O', and 'R' is to be replaced with 'ϑ-justified' and 'meet the condition $\mathcal{A}^{ex}$' with 'are to be justifiably taken to exist' if 'P', if 'P' is replaced with 'P^E'[50].

This provides a possible variant for each of the schemas $\mathbf{Sc.IA}_2^{O,P}$, $\mathbf{Sc.IA}_2^{E,P}$, $\mathbf{Sc.IA}_2^{M,P}$ and $\mathbf{Sc.SIA}_2^{O,P}$, $\mathbf{Sc.SIA}_2^{E,P}$, $\mathbf{Sc.SIA}_2^{M,P}$, without changing their respective conclusions. What may be changing in these variants are the conditions for soundness of the instances of these schemas, since

[46] In first-order language: [$^*\mathbf{Sc.IA}_2^{O/E/M,P}$] *iii.i*) $\forall y\,[\mathrm{T{\scriptstyle H}}(y) \Rightarrow [\exists x\,[\mathrm{S{\scriptstyle C}T{\scriptstyle H}}(x) \wedge P(x) \wedge \mathfrak{L}\text{-}\mathrm{I{\scriptstyle ND}}(y,x)] \Rightarrow R(y)]]$

[47] In first-order language: [$^*\mathbf{Sc.IA}_2^{O/E/M,P}$] *iii.ii*) $\forall y\,[\mathrm{T{\scriptstyle H}}(y) \Rightarrow [R(y) \Rightarrow \mathcal{A}^{ex}(a_y)]]$

[48] In first-order language: [$^*\mathbf{Sc.SIA}_2^{O/E/M,P}$] *iii.i*) $\forall y\,[\mathrm{T{\scriptstyle H}}(y) \Rightarrow [\exists x\,[\mathrm{S{\scriptstyle C}T{\scriptstyle H}}(x) \wedge P(x) \wedge \mathfrak{L}\text{-}\mathrm{I{\scriptstyle ND}}(y,x)] \Leftrightarrow R(y)]]$

[49] In first-order language: [$^*\mathbf{Sc.SIA}_2^{O/E/M,P}$] *iii.ii*) $\forall y\,[\mathrm{T{\scriptstyle H}}(y) \Rightarrow [R(y) \Leftrightarrow \mathcal{A}^{ex}(a_y)]]$

[50] If Q's are not taken to be theories, the forms of these premises have to be slightly changed, but it should not be difficult to see how this could be done in other cases.

it is possible that the instances of their premises (*iii*) be true despite the corresponding instances of the two premises in which these are split are not both true. Insofar as the conditions (*iii.i*) come to coincide with the conditions (*iii*) of the schematic arguments for veridicalism or externality—namely of $\mathbf{Sc.IA}_2^{O,V}$, $\mathbf{Sc.IA}_2^{E,Ext}$, $\mathbf{Sc.IA}_2^{M,Ext}$ and $\mathbf{Sc.SIA}_2^{O,V}$, $\mathbf{Sc.SIA}_2^{E,Ext}$, $\mathbf{Sc.SIA}_2^{M,Ext}$—the possible additional problem raised by the adoption of these variants for the soundness of genuine IA's concerns their premises (*iii.ii*), which, in case of strengthened arguments, green appear quite openly controversial.

4 Genuine IA's

> *Section summary.* In this section we offer evidence in support of the claim that no genuine sound IA has been yet delivered, and that such a (non-vacuously and non-circularly) sound argument could hardly be found.

From what has been said so far it is clear that from $\mathbf{Sc.IA}_0$ and $\mathbf{Sc.SIA}_0$ it is possible to get a large variety of schematic IA's. In order to get genuine arguments from these, one should offer so many subtle specifications that there is no hope of providing a complete inventory. This is not necessary, however. It would be enough to show that some (non-vacuously and non-circularly) sound genuine arguments (possibly at least one for platonism and one for veridicalism or externality) can be given, or else that no such argument can. We want now to offer some tentative evidence for the latter conclusion. We cannot exclude in principle that such genuine arguments can be offered. Nonetheless, it will be a burden of those who don't think our conclusion is correct green to come up with such an argument, by providing all the required determinations and by arguing for the truth of the premises thus obtained.

4.1 Strengthened IA's, again

> *Section summary.* We express doubts that a genuine sound strenghtened IA can be obtained. In order to show this, we consider different specifications of the only-if implication of $\mathbf{Sc.SIA}_0.iii$, according to different specification of the condition $\mathcal{A}$, in: both epistemic and mixed strengthened IA for externality; epistemic and mixed strengthened IA for platonism; ontological strengthened IA for both platonism and veridicalism.

Let us begin with strengthened IA's. Each of them differs from a corresponding proper IA's by including among its premises an implication that the latter lacks, namely an instance of the only-if implication of $[\mathbf{Sc.SIA}_0].iii$. No strengthened genuine IA can then be sound if no proper genuine IA is so. But even assuming that a sound proper genuine IA is available, a strengthened genuine

IA could still be unsound if the relevant instance of this supplementary implication is untrue. Let us then focus on this implication for the time being.

Replace 'Q's' with 'theories', 'the a's of some Q's' with 'the consequences of some theories', and 'meet the condition $\mathcal{A}$' with 'are ϑ-justified'. One obtains an instance of this implication that is suitable for occurring in either an epistemic or a mixed strengthened IA for externality. Consider the former option, so as to get the following schema:

[$\mathsf{\mathbf{Sc.SIA}}_2^{E,Ext}$] $iii.\Leftarrow$) The consequences of some theories are ϑ-justified only if such theories are $\mathfrak{L}$-indispensable to some scientific theories which are P^E.

We have already observed that ϑ-justification has to be an external property of the consequences of the relevant theories. This implication shows a further reason for this: if this were not the case, it would be enough to admit that there are some theories that are not $\mathfrak{L}$-indispensable to some scientific theories that are P^E to conclude that the implication is untrue. The problem is then whether one can point to an external property of the consequences of a given theory which is suitable for satisfying $\mathsf{\mathbf{Sc.SIA}}_2^{E,Ext}.iii. \Leftarrow$.

One could take it that the very fact that the relevant theories are $\mathfrak{L}$-indispensable to some scientific theories that are P^E is sufficient to bestow a specific external virtue to their consequences. If one took ϑ-justification to be such a virtue, $\mathsf{\mathbf{Sc.SIA}}_2^{E,Ext}.iii. \Leftarrow$ would be vacuously true, and this would also be the case of the inverse implication. Hence, any genuine strengthened IA whose third premise were an instance of $\mathsf{\mathbf{Sc.SIA}}_2^{E,Ext}.iii. \Leftarrow$ coupled with the inverse implication, would be vacuously sound, at best.

To get a non-vacuously sound argument, one should look for another external virtue of consequences of a given theory: a virtue distinct from their merely being consequences of the relevant kind of theory. But, for both an instance of $\mathsf{\mathbf{Sc.SIA}}_2^{E,Ext}.iii. \Leftarrow$ and the inverse implication to be true, that virtue should be just intensionally, and not extensionally, different from their being such consequences. To put it fully explicitly: in order to get a non-vacuously sound genuine strengthened IA involving an instance of $\mathsf{\mathbf{Sc.SIA}}_2^{E,Ext}.iii. \Leftarrow$, one should find a property that could be suitably taken as ϑ-justification, that could be possibly ascribed to consequences of a certain theory, and that is intensionally distinct from, although extensionally coincident with, the property of being the consequence of a theory that is $\mathfrak{L}$-indispensable to some scientific theories that are P^E.

Here we only want to suggest that this will be quite a difficult task. One could retort that naturalists can easily have such a property at their disposal. We maintain that this is not so. For present purposes, a naturalist is someone who maintains the thesis that the only form of justification that can legitimately be ascribed to a statement depends on its being a consequence of an accepted scientific theory or of a theory that is indispensable (in some appropriate way) to an accepted scientific theory. Now, if this view had to be relevant to the case at issue, acceptance of a scientific theory should depend on its having the property 'P^E' stands for, and the appropriate way in which a theory would be indispensable to a scientific theory should be the very way expressed by an appropriate determination of '$\mathfrak{L}$'. Hence, naturalism reduces here to focusing on an epistemic virtue of scientific theories, and arguing that any single statement can be justified only if it is a consequence either of a scientific theory that has this epistemic virtue, or of a theory that is $\mathfrak{L}$-indispensable to some such scientific theories. Let us notice that this thesis is at least controversial, and that many may find it in conflict with current practice in mathematics, provided that the notions of scientific and mathematical theory and of their acceptance and justification be understood as suggested in §§ 3.3 and 3.4. But this is not our main point here. Rather, the point is that this thesis gives no hint for providing an instance of $\mathsf{\textbf{Sc.SIA}}_2^{E,Ext}.iii. \Leftarrow$ that would make it and the inverse implication non-vacuously true. Accepting it, can at most lead to a sound genuine epistemic IA for externality whose conclusion is that some mathematical theorems acquire the sort of justification that comes to them from their being theorems of theories which are appropriately indispensable to some accepted scientific theories, and that this is so for the theorems of all the mathematical theories which are appropriately indispensable to some accepted scientific theories and only for them. Whereas the former claim in this conclusion is nothing but a replica of the second premise of the argument (with the only terminological addition consisting in calling 'justification' the relevant virtue), the latter is vacuously true.

The situation does not essentially change when one passes from $\mathsf{\textbf{Sc.SIA}}_2^{E,Ext}iii. \Leftarrow$ to the corresponding implications involved, respectively, in a mixed strengthened IA for externality and in an epistemic or mixed strengthened IA for platonism, that is:

[$\mathsf{\textbf{Sc.SIA}}_2^{M,Ext}$] $iii.\Leftarrow$) The consequences of some theories are ϑ-justified only if such theories are $\mathfrak{L}$-indispensable to some scientific theories which are P^0.

and

[$\mathsf{\textbf{Sc.SIA}}_2^{E/M,P}$] $iii.\Leftarrow$) The putative objects of some theories are to be justifiably taken to exist only if such theories are $\mathfrak{L}$-indispensable to some scientific theories which are P^E / P^O.

The reason is that the foregoing considerations essentially depend on the way the justification of the a's of some theories is to be understood, regardless of whether this is taken to be external justification of statements or (external) justification of the ascribed existence of putative objects, and the relevant scientific theories are taken to have an epistemic or an ontological property.

Things seem to change significantly, instead, for those instances of $\mathsf{\textbf{Sc.SIA}}_2^{E,Ext}iii. \Leftarrow$ that are involved in ontological strengthened IA's both for platonism and for veridicalism, that is:

[$\mathsf{\textbf{Sc.SIA}}_2^{O,P/V}$] $iii.\Leftarrow$) The putative objects/consequences of some theories exist/are ϑ-true if and only if such theories are $\mathfrak{L}$-indispensable to some scientific theories which are P^O.

Of course, if, in the case of ontological strengthened IA's for veridicalism, one took ϑ-truth to be a property that the relevant statements have just for their being consequences of theories that are $\mathfrak{L}$-indispensable to some scientific theories which are P^O, one would fall, *mutatis mutandis*, into a vacuity problem analogous to the one previously seen for epistemic and mixed strengthened IA's. But, if this option is discarded and existence and ϑ-truth are both taken to be conditions that putative objects and statements respectively meet or do not meet just on the basis of the way things actually are in the external world (which would result in identifying ϑ-truth with the form of truth ascribed to the relevant scientific theories in premise (i) and (ii)), another problem arises, which is even worse than this.

For it would then be hard to accept that the fact that the relevant theories are somehow indispensable to some scientific theories having some ontological property be a necessary condition for the putative objects of the former theories to exist, or for their consequences to be ϑ-true, provided these scientific theories are conceived as said in § 3.3. The reason is simply that the nature of these theories essentially depends on the contingency of the human activity that led to them. Accepting this would therefore be tantamount to claiming that from the fact that some putative objects or some statements meet a condition that they are supposed to meet or not to meet just in virtue of the way things actually are in the external world, it follows that these theories could not in fact be different from what they are like contingently, because of human activity[51]. Arguably, any instance of $\mathbf{Sc.SIA}_2^{O,P/V}.iii.\Leftarrow$ could not be true, then.

It may be possible that other ways of understanding existence and ϑ-truth would avoid these and other problems. But the burden of proof in offering such alternative conceptions would be on those willing to support ontological strengthened IA's either for platonism or for veridicalism.

Insofar as what we have said in the present section applies to any instance of $\mathbf{Sc.SIA}_0$ in which 'Q's' stands for 'theories', regardless of the way indispensability and ontological or epistemic properties of scientific theories are conceived, and any other determination of 'Q' seems to us to be nothing but a terminological variants of this, good evidence seems to have been provided that no strengthened IA's is non-vacuously sound.

[51]Clearly, most of the mathematical theories that we currently take to be indispensable to our scientific theories would not have been such before the 17$^{\text{th}}$ century, and it is absurd to think that the putative objects or statements corresponding to these mathematical theories would not have met the relevant condition then, but now do. It is easy to generalize this line of reasoning: as the well-know argument of the pessimistic meta-induction on scientific theories suggests (cf. [Laudan, 1981]), what we now take to be our best scientific theories are inductively much more likely to be false than true. One could, perhaps, envisage something as absolute indispensability, or indispensability in principle, depending on some final stage of scientific development, where all scientific and mathematical theories will be available at their ultimate level of precision and perfection. Taking this sort of indispensability to be necessary for the existence of mathematical objects or for ϑ-true of mathematical statement may not be so implausible as such. But this very notion of indispensability in principle will be highly implausible as an ingredient of IA, and contrary to how we have suggested scientific and mathematical theories should be understood to be faithful to the spirit of the argument.

4.2 Indispensability, again

Section summary. We explore how, in looking for sound genuine IA's, one can seek to obtain proper specifications of premises (*ii*) and (*iii*) according to different determinations of $\mathfrak{L}$-indispensability and conditions h and k (cf. § 3.5, above). As regards condition k, we settle on the condition of (a theory) being deductively well-balanced (cf. § 3.5). As regards condition h, we distinguish between descriptive, predictive, and explanatory tasks. These specifications deliver three modalities of indispensability: $\mathfrak{D}_\beta$-indispensability', '$\mathfrak{P}_\beta$-indispensability', and '$\mathfrak{S}_\beta$-indispensability'. These are discussed in turn in the following subsections.

Consider now genuine proper IA's, and let us admit again that 'Q's' is replaced in $\textsf{\textbf{Sc.IA}}_0$ by 'theories'.

In § 3.5, we have already noticed that any instance of premise $\textsf{\textbf{Sc.IA}}_0.ii$ seems to be untrue if it is respectively taken to assert that mathematics as a whole is $\mathfrak{L}$-indispensable to science as a whole, or that mathematics as a whole is $\mathfrak{L}$-indispensable to a part of science, and that it is all but clear that there will be true instances of this premise if it is taken to assert that part of mathematics is $\mathfrak{L}$-indispensable to science as a whole. Hence, if a genuine IA hopes to be sound, it seems that a choice among scientific and mathematical theories has to be done. Still, in order for such an argument to be sound, not only its premise (*ii*), but also its premise (*iii*) has to be true. So the relevant questions concerning a genuine IA in which the place of the Q's is taken by theories are the following: is there a scientific theory, in the sense specified in § 3.3, or a particular family of scientific theories accomplishing the same relevant task, for which a certain mathematical theory or a particular family of mathematical theories is somehow indispensable? Are these scientific and mathematical theories or families of theories such that they verify premise (*iii*) of such a genuine IA?

Answers to both questions depend on how indispensability is specified according to conditions h and k, from whose determination the determination of '$\mathfrak{L}$' in turn depends.

As regards condition k, let us suppose we settled for the condition of being deductively well-balanced (in the sense explained in § 3.5). Let us also suppose that all the scientific theories and/or their version we shall consider in what follows are deductively well-balanced. As regards condition h, consider three distinct possibilities (by far the most common ones in the discussion about IA's): suppose that this condition is either that of accomplishing a descriptive task, or a predictive task, or, finally, an explanatory task, all these tasks being conceived as part of an enquiry resulting from some sort of empirical experimental practice[52]. For short, call respectively '$\mathfrak{D}_\beta$-indispensability',

[52]It is not at all our intention to suggest that the relevant tasks are practical in nature, or limited to some pragmatic purposes. Descriptive tasks may, for example, well be identifed with the task of delivering a literally true description of some external reality. IAlso in this regard, one could evoke the notion of indispensability in principle mentioned in footnote 51 above. Even if some plausibility were given to this notion, however, appealing

'$\mathfrak{P}_\beta$-indispensability', and '$\mathfrak{S}_\beta$-indispensability' the three distinct modalities of indispensability relative to these sorts of tasks, and to the condition of being deductively well-balanced. So, concerning premise (*ii*), the relevant question is the following: is there a scientific theory or a particular family of scientific theories all accomplishing the same descriptive, predictive, or explanatory task, for which a certain mathematical theory or a particular family of mathematical theories are respectively $\mathfrak{D}_\beta$-indispensable, $\mathfrak{P}_\beta$-indispensable, or $\mathfrak{S}_\beta$-indispensable?

4.2.1 Descriptive and Predictive Indispensability

> *Section summary.* We discuss $\mathfrak{D}_\beta$-indispensability and $\mathfrak{P}_\beta$-indispensability with reference to arithmetic, real analysis, and set theory. Also by exploiting the distinction introduced in § 3.5 between single theories and families of theories, we suggest that instances of (non-vacuoulsy and non circularly) sound genuine IA involving these mathematical theories can hardly be found.

Philosophical enquiries about the ontology and semantic of mathematics are most commonly concerned with arithmetic, real analysis, and set theory. So, in considering $\mathfrak{D}_\beta$-indispensability and $\mathfrak{P}_\beta$-indispensability, let us limit our discussion these three theories or branches or mathematics[53].

One first remark is this: if set theory is conceived as a framework in which other parts of mathematics can possibly be recast, so that any theorem of these parts of mathematics can be restated as a theorem about sets, then it is hard to accept that theories belonging to these parts of mathematics, as well as the relevant fragments of set theory, are suitable for occurring in sound IA's in which 'Q's' is replaced by 'theories' and '$\mathfrak{L}$' is replaced by '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$'.

To see why, consider any version of arithmetic, let say Arith, and suppose S_i to be an instance of a scientific theory S that has recourse to an instance of Arith in which naturals numbers are

to it would be problematic for our construal of the indispensability relation only if it were admitted that such indispensability in principle is independent of the specification of any particular task. But, why should one admit this? Could one not, rather, take it that providing a final description or explanation of what the relevant scientific theories are about is a particular task, namely either a descriptive or an explanatory one? If this is conceded, as we think it should be, our charaterization of indispensability can be made perfectly compatible with this rather implausible (at least in our opinion) conception of science. It would be enough to specify the parameter 'h' in agreement with this particular task, and to consider the appropriate class of scientific theories (i.e. those at their ultimate stage) in the argument's premises.

[53] Putnam's IA given above (cf. page 11) can be seen as a paradigmatic example for IA's where $\mathfrak{D}_\beta$-indispensability is involved. Resnik's pragmatic argument can be seen as a way of focusing on $\mathfrak{P}_\beta$-indispensability. Baker's version of IA (cf. page 20, above) is clearly based on $\mathfrak{S}_\beta$-indispensability (as is the version of IA suggested by [Field, 1989], p. 15) and will be discussed in more details below. Colyvan's argument (cf. page 19, above) is not explicitly relying on a particular notion of indispensability: this seems to us partly due to the fact that the notion of indispensability occurs in it in a schematic way, that should be then made precise according to different determinations. In [Panza and Sereni, 2015] we suggest how the versions of IA commonly discussed can be retrieved by determination of schematic premises from schematic IA's cognate to the ones discussed here.

taken to be *sui generis* objects and no appeal is made to set theory. There are, of course, countless possible examples of this. It is highly plausible to admit that there is another instance S_j of S, or an instance S_j^* of another scientific theory S^* accomplishing the same descriptive or predictive task as S, both of which have no recourse to Arith but rather to an appropriate fragment of set theory in which Arith is recast. In the former case, Arith is neither $\mathfrak{D}_\beta$-indispensable nor $\mathfrak{P}_\beta$-indispensable to S, which is already enough to block premise (*ii*) of any genuine IA where 'Q's' is replaced by 'theories', '$\mathfrak{L}$' is replaced by '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$' and the relevant scientific and mathematical theories would respectively be S and Arith. In latter case, Arith is possibly $\mathfrak{D}_\beta$-indispensable and/or $\mathfrak{P}_\beta$-indispensable to S, but not for any family of scientific theories including both S and S^*. Hence, insofar as S and S^* share the same descriptive or predictive task, it follows that this task can be accomplished even though no recourse to textsfArith is made. This suggests that, in this latter case, a genuine IA where '$\mathfrak{L}$' is replaced by '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$', 'Q's' by 'theories', and the only relevant mathematical theory is Arith, would be unsound, both if S is the only relevant scientific theory, and if the relevant scientific theories are those of a family of such theories including S and S^*. In the former case, premise (*ii*) would be untrue; in the latter, premise (*iii*) would be untrue.

The same argument can be repeated with respect to the relevant fragment of set theory (understood, again, as a framework in which other mathematical theories can be recast), if one inverts the roles of this particular fragment of set theory and Arith in the reasoning above. Hence, the same conclusions as above seem to hold if Arith is replaced by this fragment of set theory.

It still remains that, on the basis of the definitions advanced in §3.5, the two cases considered are such that the family of mathematical theories composed of Arith and the relevant fragment of set theory—call this family 'AST', for further reference—is possibly $\mathfrak{D}_\beta$- or $\mathfrak{P}_\beta$-indispensable to S and to the family of scientific theories composed of S and S^*, respectively. Suppose this is so, and suppose also that S and S^* both have an appropriate property P. It follows that premise (*ii*) of a genuine IA appropriately specified would be true in both scenarios. But is this also the case for premise (*iii*)? Two cases are to be distinguished here.

The first occurs if the expression 'their a's' in premise (*iii*) of $\mathbf{Sc.IA}_0$ is taken to refer conjunctively to the a's of the two theories composing AST. It seems clear that, in this first case, this premise would be untrue, regardless of the determinations of 'a' and '$\mathcal{A}$', since, even if AST were $\mathfrak{D}_\beta$ or $\mathfrak{P}_\beta$-indispensable to either S or to the family of theories composed of S and S^*, neither Arith nor the relevant fragment of set theory would be, if taken individually.

The second case occurs if the expression 'their a's' in premise (*iii*) of $\mathbf{Sc.IA}_0$ is taken to refer disjunctively to the a's of the two theories composing AST. In this second case, the reasons for thinking, as it happened in the first case, that this premise is untrue are no longer enough to support this conclusion. It does not seem to us, however, that there is any easily available argument supporting the opposite conclusion that this premise would be true. Any supporter of IA willing to endorse such an argument in a form appropriate to a similar case would have the burden of providing such an argument. To make the situation clearer, let us suppose 'P', 'a', and

'$\mathcal{A}$' to be so determined that this premise came to assert that, if AST is $\mathfrak{D}_\beta$ or $\mathfrak{P}_\beta$-indispensable to S or to the family of scientific theories composed of S and S^*, both of which are true, then either the objects of Arith or those of the relevant fragment of set theory exist. Clearly, in order for the premise to be true, it would be enough that either the objects of Arith or those of the relevant fragment of set theory exist. But supposing that this is so would clearly beg the question with respect to the relevant genuine IA, and claiming that this is what makes this premise true would inescapably make such an IA circular. The point is rather whether the truth of the antecedent of this implication is a sufficient reason for its consequent to hold. This is what the argument to be provided should argue for. And, *mutatis mutandis*, this would also be the case if 'P', 'a', and '$\mathcal{A}$' were determined in some other appropriate way.

It seem to us that such an argument could be hardly advanced. Still, let us suppose that it is available. There would, then, be room for concluding that a genuine IA, whose premise (*iii*) refers disjunctively to the a's of the two theories composing AST, is non-vacuously sound. But, then, its conclusion would be that either the a's of Arith, or those of the relevant fragment of set theory, meet the condition $\mathcal{A}$, for an appropriate determination of 'a', and '$\mathcal{A}$'. And this would be at odds with the spirit underlying IA's, according to which we want to establish that certain mathematical a's meet the condition $\mathcal{A}$ because the corresponding theories are somehow indispensable to some scientific theories, and not that some or other mathematical a's meet this condition because one or another among the corresponding theories are somehow indispensable to some scientific theories. This seems confirmed by how considerations on alternative foundations for mathematics, like those offered by [Baker, 2003] (by developing suggestions from [Benacerraf, 1965]), are usually regarded in the context of discussions on IA: in these cases it is indeed usually argued that the possibility of having different incompatible and still in some sense mathematically equivalent foundations for classical mathematics (e.g. through either set theory or category theory) is a reason for denying that IA can provide justification for the commitment in any of the objects of those theories in particular, and hence to deliver what it is supposed to deliver[54].

Moreover, even if one were content with settling for an IA with a minimal anti-nominalist conclusion IA as the foregoing[55], one would end up supporting a very weak form of platonism,

[54]Compare what [Baker, 2003], p. 58, claims:

> Category theory is not an extension of set theory, nor *vice versa*, and the ontologies of the two theories are entirely non-overlapping. Thus neither set theory nor category theory is indispensable for science, because neither provides a unique foundation for mathematics. Hence we are not rationally compelled to believe in the existence of sets, nor are we rationally compelled to believe in the existence of categories. Our ontological commitment to mathematical objects cannot be made more specific than a disjunctive commitment to sets-or-categories. [...] Commitment to a specific ontology of abstract objects cannot be derived from the indispensability argument alone.

[55]That some such minimal anti-nominalist conclusion may be considered by some in the context of an IA based

veridicalism, or externality, and indeed one rather alien to most current versions of these views. Consider platonism. The purpose of most of its current versions is not merely to secure some weak ontological claim concerning the existence of some mathematical objects, but also to explain how we can have epistemic access to these objects—something that seems impossible if one is not even able to tell with precision which objects one is speaking of. True, many formulations of IA's feature a very general conclusion to the extent that there exist mathematical objects, or we are justified in believing they do. But as we have argued before, these arguments should be merely considered as schematic, and this makes it difficult to assess whether they are, in some sense, sound or not. If we had no instance of such an argument, we would then be unable to assess whether the conclusion is well supported or not. Moreover, it seems to us that the common understanding of that conclusion is as relating to particular instances of such arguments, and thus to particular mathematical objects (as opposed to some objects or other). This seems consistent with the general attitude of supporters of IA (see again the end of Baker's quotation in footnote 54).

Let us now come back to our main discussion. The arguments given above cannot be repeated in the case where '$\mathfrak{L}$' is replaced by '$\mathfrak{S}_\beta$' and the relevant descriptive and predictive tasks are replaced by an explanatory task. The reason is that it is not plausible at all to admit that, in the case where S_i is an instance of a scientific theory S that has recourse to a deductively well-balanced instance of Arith in which natural numbers are taken to be *sui generis* objects, there is another instance S_j of S, or an instance S^*_j of another scientific theory S^*, accomplishing the same explicative task as S, that have not recourse to Arith but rather to some fragment of set theory in which Arith is recast. Indeed, while it is plausible that the replacement of Arith with this fragment of set theory will preserve the descriptive and predictive powers of the relevant scientific theory, it is far less plausible that it will preserve its explicative power, since this power may essentially depend on the specific way in which natural numbers are defined in Arith. For example, if we suppose Arith is Frege Arithmetic, one could argue that its defining natural numbers as numbers of concepts allows one to appeal to it for a crystal-clear explanation of why there are as many animals that are either dogs or cats as there are mammals in a zoo where there are only dogs, cats, crocodiles and eagles, whereas this explanation cannot be (as clearly) given by appealing to any rephrasing of natural numbers within pure set theory (where natural numbers are, for example, coded with Zermelo's sets like $\{\ldots\{\emptyset\}\ldots\}$). In the next section we will consider other reasons for doubting that any genuine IA based of '$\mathfrak{S}_\beta$'-indispensability can be given.

Before coming to that, let us change our perspective, and consider mathematical theories, independently of the possibility of rephrasing them within set theory, this latter theory being considered in turn as an autonomous mathematical theory, specifically concerned with sets as *sui generis* objects. With respect to epistemic IA's, however, an argument similar to the previous one can also be advanced for arithmetic and real analysis.

on explanatory indispensability is considered by [Sereni, 2015]'s rejoinder to [Molinini, 2015], in this volume.

Consider, for example, any version RealAn of real analysis and suppose that S_i is an instance of a scientific theory S that has recourse to an instance of RealAn. It is highly plausible to admit that there will be another instance S_j of S, or an instance S_j^* of another scientific theory S^* that has no recourse to RealAn, insofar as it replaces it with an appropriate theory of rational numbers—that is, with an appropriately extended version of arithmetic—, which accomplishes the same descriptive or predictive task as S_i in such a way that no difference among the two, with respect to the accomplishment of this task, can be appreciated on the basis of our capacity of discerning their descriptive and/or the predictive power (on the basis of which we assign to these theories the epistemic property P^E). Hence, if S is P^E, also S^* is so, and there are good reasons for thinking that the conditions h from which $\mathfrak{D}_\beta$-indispensability and $\mathfrak{P}_\beta$-indispensability respectively depend are such that, whatever S might be, RealAn is neither $\mathfrak{D}_\beta$-indispensable nor $\mathfrak{P}_\beta$-indispensable to S, or it is possibly $\mathfrak{D}_\beta$-indispensable and/or $\mathfrak{P}_\beta$-indispensable to S but not to a family of scientific theories including both S and S^*. From here, the argument can go on on the same lines as above by replacing Arith and its recasting within set theory with RealAn and the appropriate theory of rational numbers.

The same argument does not apply in the case of ontological or mixed IA's, for the following reason. For such an IA to be correct, the scientific theory S it involves must have a certain ontological property P^O. Now, one could argue that replacing real numbers with rational ones in S_i inevitably results in passing from S to a new theory S^* which is not P^O any more. Suppose that P^O is (external) truth (as defined earlier). One could argue that S_i's having recourse to RealAn is essential for it to be an instance of a(n externally) true theory, so that replacing RealAn with a theory of rational numbers transforms S_i in an instance of another theory S^* that is not (externally) true any more. Hence, whatever S^* might be, it cannot accomplish any appropriate relevant task that S accomplishes, since for this to be the case S^* has to be (externally) true true like S is.

So far so good. But how is it possible to claim that when one replaces real numbers with rational numbers in (an instance of) a true scientific theory, this will inevitably result in getting a new, (externally) untrue theory, if not by admitting (either explicitly or implicitly) that real numbers are part of what makes this theory (externally) true? Once this is admitted, there is no room for getting a non-circularly sound genuine ontological or mixed IA involving any version of real analysis. Such a genuine IA could perhaps be sound and even non-vacuously so, but it will inevitably be circular, and then, at most circularly sound. The same argument applies, of course, *mutatis mutandis*, for any ontological or mixed IA involving real analysis, that is for any other determination of P^O.

There seems to be little hope, then, of getting a non-circularly sound genuine IA's involving real analysis in which 'Q's' is replaced by 'theories' and '$\mathfrak{L}$' is replaced by '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$'.

For (finitary) arithmetic things look even simpler. In order to show that there cannot be any sound genuine IA's involving (finitary) arithmetic in which 'Q's' is replaced by 'theories' and

'$\mathfrak{L}$' is replaced by '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$', it is enough to notice that natural numbers can be replaced with numerical quantifiers in any scientific statement in which they occur, without loosing or weakening the descriptive and predictive power of this statement.

There seems to be no analogous argument for set theory conceived as an autonomous mathematical theory specifically concerned with sets. It is not clear, however, that there is any suitable argument available suggesting that a genuine IA based on descriptive or predictive indispensability and concerned with set theory so conceived is forthcoming[56]. However, one could argue along the following lines. Let us consider the role that could be played by set theory, conceived as an autonomous theory, within a given scientific theory S, relative to its descriptive and predictive power—that is, let us concern ourselves with '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$'-indispensability. It can be argued that its role amounts, globally, to the role that could be played, piecemeal and conjunctively, by other mathematical theories, also conceived as autonomous. For instance, a sub-system of set theory will play the role of arithmetic, another will play the role of function theory, another will play the role of real analysis, and so on. If an autonomous mathematical theory (independently of the fact that it could in principle be rephrased within set theory) can be found to play the role of each of these sub-systems of set theory, it would follow from what we have said above that no particular fragment of set theory among these will be by itself either '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$'-indispensable to S, nor to a family of theories accomplishing the relevant task that S accomplishes. One could then argue by this piecemeal reasoning that no genuine, non-vacuously sound IA for the '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$'-indispensability of set theory by itself will be available[57].

Of course, what we have said does not necessarily extend to any possible genuine IA based on $\mathfrak{D}_\beta$-indispensability or $\mathfrak{P}_\beta$-indispensability, and *a fortiori* any possible genuine IA based on a

[56]This is not to say, of course, that some known versions of IA could not be read as such. For one, Quine's long exposition in chapter 5 of [Quine, 1960] can easily be read, and has indeed been read, as outlining a version of indispensability argument for classes. The main reason offered by Quine for being committed just to classes, rather than, say, to classes and natural numbers, is that classes neatly allow for "the explication of the various other sorts of abstract objects" (p. 267). This would make Quine's argument irrelevant in the present context, where we are looking for arguments based on the descriptive or predictive power of set theory beyond the fact that it allows for the reformulation of other mathematical theories. Quine also adds that "the power of the notion [of class] on other counts, [...] keeps it in continuing demand in mathematics and elsewhere as a working notion in its own", so that it "confers a power that it is not known to be available through less objectionable channels". These additional reasons would be relevant here only if it turned out that they did not depend on the power of set theory (the notion of class) to systematize and unify different (mathematical and non mathematical) theories by allowing for their reformulation in terms of classes. Here we just want to stress that it is not so clear that this is the case. Therefore, it is all but clear that Quine's remarks provide an argument that a genuine IA based on descriptive or predictive indispensability and concerned with set theory as such is forthcoming.

[57]In this scenario, it would still be possible for set theory to be part of a family of mathematical theories which, as such, are '$\mathfrak{D}_\beta$' or '$\mathfrak{P}_\beta$'-indispensable to a given scientific theory, or to a family of such theories; one could, then, argue that some genuine, non-vacuously and non-circularly sound IA concerned with this family of mathematical theories, whose premise (*iii*) is intended disjunctively, as explained above at pages 38-39, may be available. Here, however, the same considerations raised there apply too.

non-explanatory form of indispensability. However, the preceding considerations seem to raise a significant concern on the actual possibility of attaining such a non-circular and non-vacuoulsy sound argument.

4.2.2 Explanatory Indispensability

> *Section summary.* We explore attempts to find sound genuine IA's based on the notion of $\mathfrak{S}_\beta$-indispensability. We consider the role that Inference to Best Explanation can have in supporting versions of IA based on this modality of indispensability, and discuss Baker's Enhanced Indispensability Argument. We argue that in all versions of IA envisioned so far, an appeal to $\mathfrak{S}_\beta$-indispensability relies on claims that are highly dubious or on question-begging readings of premise (*iii*). Also by relying on the notions and distinctions introduced in the previous sections, we again submit that we have not yet been presented with an instance of a sound genuine IA based on explanatory indispensability, and offer evidence that arguments of this sort can hardly be found.

For IA's based on explanatory indispensability, things are different.

Let us begin by considering epistemic IA's for externality. Starting from $\textbf{Sc.IA}_0$, replace again 'Q's' with 'theories', and 'P' with 'P^E', '$\mathfrak{L}$' with '$\mathfrak{S}_\beta$', 'the a's of some Q's' with 'the consequences of some theories', 'meet the condition $\mathcal{A}$' with 'are ϑ-justified', so as to get instances of its three premises suitable for such an IA, namely:

[$\textbf{Sc.IA}_3^{E,Ext,Exp}$]

i) Some scientific theories are P^E.

ii) Among them, some are such that some mathematical theories are $\mathfrak{S}_\beta$-indispensable to them.

iii) If some theories are $\mathfrak{S}_\beta$-indispensable to some scientific theories which are P^E, then their consequences are ϑ-justified.

Of course, one could take here ϑ-justification to be a property that the consequences of certain theories have just insofar as these theories are $\mathfrak{S}_\beta$-indispensable to some scientific theories that are P^E. But this would lead to a problem analogous to that advanced for strengthened epistemic IA's in relation to the analogous option for that case. Differently from what we have seen above for strengthened IA's, however, no problem would be engendered by taking ϑ-justification to be an external property possibly enjoyed by the consequences of some theories that are not $\mathfrak{S}_\beta$-indispensable to some scientific theories that are P^E (namely a property that these theories would have beyond their being consequences of certain theories).

To make things clearer, suppose that this property is a property that a statement has just in case it presents itself as true to our rational consideration, because of what it says, and of some circumstances which are independent of its being a consequence of a given theory. Let us say, that a statement having this property presents itself as true. This property is, by definition, independent of the relevant statements being consequences of some theories that are $\mathfrak{S}_\beta$-indispensable to some scientific theories that are P^E. Nonetheless, their being so may help bestow this property to them. Premise $\mathsf{\mathbf{Sc.IA}}_3^{E,Ext,Exp}.iii$ becomes then the following:

[$\mathsf{\mathbf{Sc.IA}}_4^{E,Ext,Exp,PrT}$] *iii)* If some theories are $\mathfrak{S}_\beta$-indispensable to some scientific theories which are P^E, then their consequences presents themselves as true.

The question is then whether $\mathfrak{S}_\beta$-indispensability is such that being $\mathfrak{S}_\beta$-indispensable to some scientific theories with property P^E is a sufficient condition for a theory to present itself as true.

This is likely to depend on the way $\mathfrak{S}_\beta$-indispensability is determined. Still, it is reasonable to suppose that, granting a positive answer is possible at all, it will be forthcoming in a case in which $\mathfrak{S}_\beta$-indispensability depends on an occurrence of inference to the best explanation (IBE). Let us, then, focus on this case, and confine ourselves to the simplest sub-case in which $\mathfrak{S}_\beta$-indispensability is supposed to relate single theories to single theories. Take then a theory T to be $\mathfrak{S}_\beta$-indispensable to a scientific theory S if and only if any instance of S which provides the best explanation of some relevant accepted claims E's has somehow recourse to an instance of T[58]. This entails that any theory T is $\mathfrak{S}_\beta$-indispensable to a scientific theory S no instance of which provides the best explanation of the relevant accepted claim E's. Still, for this determination of $\mathfrak{S}_\beta$-indispensability to satisfy the requirements advanced in § 3.5, such a scientific theory S cannot be P^E in this case. Suppose then that, for a scientific theory S to be P^E, it has to provide the best explanation for some accepted claims E's pertaining to its range of application, and that another theory T is $\mathfrak{S}_\beta$-indispensable to such a scientific theory if and only if any instance of S which provides such a best explanation of claims E's has somehow recourse to an instance of T (supposing, of course, that a scientific theory provides the best explanation of some claims E's if and only if some instances of it, but not necessarily all of them, do). The question thus becomes the following: is it sufficient to acknowledge that a scientific theory S is P^E and that any instance of it which provides the best explanation of the relevant claims E's has appropriately recourse to an instance of another theory T, in order to conclude that the consequences of T present themselves as true?

Consider the most famous example that has been advanced with the aim of offering a positive answer, Baker's example of a particular subspecies of periodical cicadas. The relevant scientific theory S is here some fragment of evolutionary biology dealing with the life cycle of some species of periodical cicadas of the Magicicada genus, while E is the claim that the adult cicadas of these species emerge from the nymphal stage "after either 13 years or 17 years depending on the geographical area" ([Baker, 2005], p. 229). Call the former '$\mathsf{EvBiolCic}$', and the latter '$\mathsf{13\text{-}17LC}$'.

[58] This means, of course, that the relevant specification of the parameters h and k capture what it is for a theory to provide the best explanation of the relevant accepted claims E's.

This example becomes relevant for the present discussion as soon as one claims that arithmetic, or at least some appropriate fragment of it, or some version of it, or some fragment of such an appropriate version of it, are $\mathfrak{S}_{\beta}$-indispensable to $\mathsf{EvBiolCic}$ since the best explanation of $\mathsf{13\text{-}17LC}$ requires an appeal to the statement that 13 and 17 are prime numbers, as well as to a simple arithmetical theorem about primes numbers, and that this entails that any instance of $\mathsf{EvBiolCic}$ which provides such a best explanation has necessarily recourse to an instance of arithmetic or of the relevant fragment of it. There may be several reasons for doubting that this is so. Still, for the sake of our present argument, let us concedes this, and call 'Arith^*' either arithmetic as such, or the relevant fragment, or version, or fragment of a version of it. What we have to ask is then whether this is enough for concluding that the theorems of Arith^* present themselves as true. Whatever Arith^* might be, it seems obvious that its theorems should include the relevant theorem about prime numbers, or at least a restriction of this theorem to the small prime numbers the consideration of which is enough for delivering the relevant explanation[59].

So, we can confine ourselves to considering whether Arith^*'s alleged $\mathfrak{S}_{\beta}$-indispensability for $\mathsf{EvBiolCic}$ is enough for concluding that this theorem, even thus restricted, presents itself as true.

It should be borne in mind that, in the context of the present discussion, the property of presenting itself as true has to be understood as an external property of that theorem. It seems, then, that in order to get a positive answer to the question above, one must claim that what makes Arith^* $\mathfrak{S}_{\beta}$-indispensable to $\mathsf{EvBiolCic}$ (that is, what makes it the case that any instance of $\mathsf{EvBiolCic}$ which provides the best explanation of $\mathsf{13\text{-}17LC}$ will somehow recur to an instance of Arith^*) is not merely the fact that appealing to prime numbers (or maybe only to small ones among them) and to their properties is needed for the relevant best explanation to be stated, but rather that prime numbers themselves (or possibly small ones among them), or at least some sort of circumstances involving these numbers themselves, provide such a best explanation. Now, it is easy to see that this is not so at all. Indeed, the relevant content of $\mathsf{13\text{-}17LC}$ for the purpose of explanation—that is, what has to be explained—is not properly that adult periodical cicadas emerge from the nymphal stage after either 13 or 17 years, but rather that they emerge from the nymphal stage when they actually do: we appeal to 15 and 17 in order to describe a fact which

[59]Pincock objects to Baker (cf. [Pincock, 2012], pp. 212-213) that his alleged explanation fails what Pincock calls the 'weaker alternatives problem': for an agent that does not already believe that there are infinitely many primes, there seems to be no reason for preferring an explanation based on the claim that prime periods minimize intersections with predators' life-cycles, rather than an explanation based on the claim that prime periods less than 100 years minimize intersection. Hence, there is no reason for accepting, on the basis of an application of IBE to the cicada case, that there exist infinitely many prime numbers, rather that merely those less than 100. The point seems well taken in the framework of Pincock's discussion, and is reminiscent of a similar point raised by [Peressini, 1997, pp. 220-223] as the problem of "the unit of indispensability". However, we believe it is possible to object to Baker's argument even if it is admitted that one can decide whether the theory to be taken into account in the case at issue is arithmetic as such, or a fragment of it only dealing with small natural numbers, for example with those less than 100. Hence, for the sake of our present argument again, we take for granted that a decision between these two alternative theories is possible.

is, in itself, perfectly independent of the existence and properties of any number and could in principle be described without any appeal to those numbers, or possibly to any number at all[60]. And what provides the alleged best explanation is nothing but the circumstance that the life cycle of periodical cicada is such as to minimise the "frequency of intersection" with the life cycle of hypothetical periodic predators or of hypothetical similar subspecies suitable for hybridisation ([Baker, 2005], pp. 230-31), which also is perfectly independent of the existence and properties of any sort of number. Arguing that Arith^* is $\mathfrak{S}_\beta$-indispensable to $\mathsf{EvBiolCic}$ merely reduces to arguing that the appeal to prime numbers (or possibly just to small ones) and to their relevant properties is required in order to state such a best explanation; for example, it is required in order to give a clear description of this non-mathematical circumstance to someone already familiar with arithmetic or a fragment of it. In other terms, Arith^* is at most an indispensable tool to be used by evolutionary biologists, in the present historical conjuncture, in stating the best explanation of $\mathsf{13\text{-}17LC}$. But this will not certainly suffice to ensure that the relevant theorem of Arith^* presents itself as true.

But this is not all, yet. What is even more relevant for our present purpose is that admitting that what makes Arith^* $\mathfrak{S}_\beta$-indispensable to $\mathsf{EvBiolCic}$ depends on the existence and properties of prime numbers themselves (or possibly just on those of small ones) would undoubtedly beg the question against the anti-platonist opponent, and make the relevant IA's circular. For this would require acknowledging that Arith^* is actually about natural numbers, and says of them that they meet some conditions relevant for their playing, by themselves, an explanatory role; and this entails that Arith^* tells us something true of these numbers. In order to conclude that Arith^*, or at least some of its theorems, presents itself as true, one should then already acknowledge that they are actually true.

The point can easily be generalised. For an instance of premise $\mathbf{Sc.IA}_4^{E,Ext,Exp,PrT}.iii$ to be plausibly taken to be true for a scientific theory S, under the supposition that $\mathfrak{S}_\beta$-indispensability relates single theories to single theories, it must be admitted that what makes T $\mathfrak{S}_\beta$-indispensable to S is not merely that having recourse to T is needed for the best explanation of claims E's to be stated; rather, this best explanation should be assumed to be provided by that which T is taken to be about, or at least by some sort of circumstances involving that which T is taken to be about. But this assumption begs the question, and makes the relevant IA's circular. Moreover, if we are considering the case in which T is a mathematical theory, this can hardly be the case, at least if mathematical theories are understood as suggested in § 3.4. For these theories can hardly be

[60] A way of providing such a number-free description has been suggested by [Rizza, 2011]. [Baker, 2009], pp. 618-622, had previsously suggested that such a description is not possible, since mathematical terms such as 'prime' cannot be eliminated from the statement expressing the *explanandum* in the periodical cicadas case. Notice, in passing, that Baker there also answers to the challenge raised by [Bangu, 2008] according to which he would be begging the question against the nominalist, since the explanandum must be formulated in mathematical terms (and obviously believed true).

taken to speak of things that can provide in themselves, or be involved in themselves, in some sort of circumstances which provide the best explanation of some accepted claims E's that a scientific theory could be plausibly be required to explain. Such claims E's cannot but concern an enquiry resulting from some sort of empirical experimental practice. And it seems hard to concede that an actual mathematical theory, a mathematical theory actually involved in scientific practice, could be taken, without begging the question, to speak of things that can plausibly provide in themselves, or be involved in themselves in some sort of circumstances which provide, the best explanation of such claims. At most, one can acknowledge that an actual mathematical theory is such that appealing to what it putatively is about is required for describing, in a perspicuous way, what actually provides the best explanation of such claims to peoples familiar with a certain mathematical background.

One can even push this objection further, and object that it is even question-begging to maintain that the fact that a mathematical theory is indispensably appealed to in what we take to be the best explanation of some scientific phenomena can, by itself, justify that mathematical theory. Something along these lines has been suggested by [Pincock, 2012], (§§10.2-10.3)[61]. Pincock's point, in a nutshell, is that "mathematical claims must be justified prior to their use in explanation if they are to make their explanatory contribution" (p. 217). In other words, a scientist that is offering a given explanation of some phenomena will have to antecedently believe the mathematical claims (s)he is employing to be true (or at least, we might add, appropriately justified), if (s)he is to understand her/his own explanation at all. If (s)he has no way of explaining how scientists can be justified in their mathematical beliefs "prior to their use in science", there is little hope "that some kind of indispensability argument could do better" ([Pincock, 2012], pp. 219-220). This is, at least, what follows from the account of the application of mathematics that Pincock adopts. Another alternative account could lead to a different, or even opposite, conclusion. Still, it remains that the mere plausibility of Pincock's account suggests that a genuine non-circular IA based on explanatory indispensability requires an appropriate account of the application of mathematics, alternative to Pincock's, that a supporter of such an argument is, then, required to offer.

A way of avoiding the difficulty could be to imagine a scenario in which a genuine epistemic IA's based on explanatory indispensability work in a context in which other epistemic IA's, not based on explanatory considerations, have conferred justification to some mathematical beliefs. Indeed, in a proper (as opposed to a strengthened) epistemic IA, nothing requires that indispensability for appropriate scientific theories is the only source of justification for mathematical statements. There is then no obstacle in imagining genuine epistemic IA's based on explanatory indispensability working in a context in which some mathematical beliefs have been otherwise justified. In this case, the relevant genuine epistemic IA's would merely enhance or extend the

[61]Pincock makes his point in connection with the adoption of a contrastive account of IBE. Our present remarks are indipendent of the adoption of such an account.

already available justification for mathematical beliefs. This would make such explanatory IA's much less useful, or not privileged as sources of justification, at worst; but Pincock's argument would make them neither circular nor unsound.

The aim of our previous considerations is to rise serious doubts on the possibility of having such an IA. Suppose, indeed, that the best explanation of certain claims has been identified (by whatever account of IBE), and that it is such that a certain mathematical theory is $\mathfrak{S}_\beta$-indispensable to a scientific theory which is P^E. If our considerations above are correct, it would still be the case that taking an instance of premise $\textbf{Sc.IA}_4^{E,Ext,Exp,PrT}.iii$ to be true would beg the question (so as to make the corresponding IA's circular), and be highly doubtful.

It is not difficult to see that the same considerations also apply (without any modification) to epistemic IA's for platonism whose third premise is an instance of the following schema, under the admission that $\mathfrak{S}_\beta$-indispensability is supposed to relate single theories to single theories:

[$\textbf{Sc.IA}_3^{E,P,Exp}$] *iii)* If some theories are $\mathfrak{S}_\beta$-indispensable to some scientific theories which are P^E, then their putative objects are to be justifiably taken to exist.

Indeed, in order to argue that an instance of this premise is meaningful (and, *a fortiori*, in order to argue that it is true), one has to admit that the former theories do have some putative objects. This is tantamount to claiming that these theories speak of these objects. Hence, what has been said above concerning what the theory T is taken to be about directly applies to them: for an instance of premise $\textbf{Sc.IA}_3^{E,P,Exp}.iii$ to be true, the reason for claiming that the former theories (each of them being individually taken) are $\mathfrak{S}_\beta$-indispensable to the latter (each of them being also individually taken) has to depend on the relevant best explanation's being provided by the very objects that the former theories are supposed to be about, and this both begs the question and is highly dubious.

The same considerations can also be easily extended to the case where $\mathfrak{S}_\beta$-indispensability is supposed to relate families of theories to single theories, or single theories to families of theories, or families of theories to families of theories, and, with slight modifications, also to other sorts of epistemic IA's based on $\mathfrak{S}_\beta$-indispensability both for externality and for platonism. Looking at the argument, it is also easy to see that it does not essentially depend on the assumption, which we have made so far, that $\mathfrak{S}_\beta$-indispensability depends on an occurrence of IBE. Hence, one can reach the same conclusion also by maintaining (on the basis of some argument which we are not able to imagine) that $\mathfrak{S}_\beta$-indispensability goes without IBE.

The situation is certainly not better for ontological and mixed IA's based on explanatory indispensability. Replacing an epistemic property P^E with an ontological property P^O does not change the situation for the third premise of an IA's, both in the case of arguments for veridicalism or externality and for platonism, and whatever condition '$\mathcal{A}$' may stands for. In all these cases, for an instance of this premise to be true, in the case where the Q's are taken to be theories, the alleged $\mathfrak{S}_\beta$-indispensability has to depend on the fact that the relevant explanation (be it the best or not) is provided by that which these theories are about, so that the point made above still applies.

5 Conclusions

Our aim, in § 3, has been that of providing a systematic and exhaustive account of all possible varieties of IA's, both proper and strengthened. Each of the latter includes one of the former as a sub-argument. The former are intended to offer sufficient conditions for their conclusions. The latter offer necessary and sufficient ones. These conclusions can be of different nature, but they all concern either some sort of mathematical statements (most usually theorems) or mathematical entities or putative objects. Statements occur either in IA's for veridicalism, the thesis that these statements are true, or for externality, the thesis that these statements have an epistemic virtue external to the theory to which they belong to, or, more in general, to the justificatory system from which they result. Entities or putative objects occur in IA's for platonism, the thesis that there exist objects that mathematics is about. IA's for veridicalism are all ontological, that is, they are based on the assumption that some scientific theories have an appropriate ontological property, for example (external) truth. IA's for externality are all epistemic, instead, that is, they are based on the assumption that some scientific theories have an appropriate epistemic property, for example they are rationally justified. Among IA's for platonism some are ontological, other epistemic.

The purpose of this account is not only, and not mainly, to offer a complete synopsis of all possible IA's. Our main purpose has rather been to show that providing a genuine IA, that is, a particular fully determined version of the argument, is far from easy. So far from easy is it that, at the best of our knowledge, no such an IA has been offered yet. What have rather been offered are several schematic IA's, that is, arguments schemas waiting for future determinations. Argument schemas are valid or invalid, and most of those that have been offered are valid, indeed. But, in general, argument schemas are neither sound nor unsound: genuine IA's are. Hence, we submit that, for the most part, recent discussions about the soundness or unsoundeness of the different IA's actually available miss their target, since they concern the truth or untruth of essentially indeterminate premises. According to us, before discussing whether these premises are (plausibly) true or untrue, one should clearly determine them, that is, make clear what they are asserting exactly. This is especially so for the premises or the parts of them which are concerned with the relation of indispensability. Often, in the context of an IA, one argues or denies that something is indispensable to something else, or that the indispensability of something for something else entails this or that consequence, without making clear not only what it means that something is indispensable to something else, but also what is supposed to be indispensable to what. In §§ 3.4 and 3.5, we have tried to account for all the specifications which are required in order to make such claims determinate, and thus possibly true or untrue.

Our aim, in § 4, has been to suggest that no genuine IA has a chance to be sound, unless vacuously or circularly. It goes without saying that we have not examined all possible genuine IA's. This would have been virtually impossible. And this not so much because of the intractable length and complexity of such an enquiry, but rather because of the fact that this would have required considering a number of possible determinations of the schematic IA's listed in § 3 that we are quite far from being able to imagine. Our strategy has rather been that of identifying different constraints that the determinations required in order to transform schematic into genuine IA's have to respect, and then arguing against the possibility of respecting all of them at once without getting either circular or vacuously sound arguments. To this purpose, we have focused on some possible schemas of premises, by arguing that no appropriate instance of them can provide true premisses for a non-circular and non-vacuously sound argument. This has led us to offer a number of sufficient reasons for discarding the possibility of non-circular and non-vacuously sound IA's of different sorts. But we are far for thinking that we have, in this way, expounded all possible reasons by which a genuine IA can be unsound or circular or only vacuously sound. Many other possible reasons may be envisaged, for example some that would undermine the truth of the different instances of the first premise of ontological IA's of any sort.

Hence, even if we do not take ourselves to have offered conclusive reasons against the possibility of offering a non-circular and non-vacuously sound genuine IA, it is still an outcome of the foregoing discussion that in order to provide one, it is by far not enough to refute one or more of the arguments we presented above. It will also and overall be necessary to provide all the determinations suitable for transforming a schematic IA in a genuine IA, and then to argue with due care for its non-circularity and non-vacuous soundness.

Many supporters of veridicalism, externality, or platonism might be willing to get a non-circular and non-vacuously sound argument for their views. We clearly do not take our arguments to go against those views themselves, nor do we believe we have offered any evidence against them. We have only tried to show, defeasibly as this could be done, that they can hardly be supported by some forms of IA.

References

[Baker, 2003] Baker, A. (2003). The indispensability argument and multiple foundations for mathematics. *Philosophical Quarterly*, 53(210):49–8211.

[Baker, 2005] Baker, A. (2005). Are there genuine mathematical explanations of physical phenomena. *Mind*, 114:223–238.

[Baker, 2009] Baker, A. (2009). Mathematical explanation in science. *British Journal for the Philosophy of Science*, 60:611–633.

[Bangu, 2008] Bangu, S. I. (2008). Inference to the best explanation and mathematical realism. *Synthese*, 160(1):13–20.

[Benacerraf, 1965] Benacerraf, P. (1965). What numbers could not be. *Philosophical Review*, 74(1):47–73.

[Bueno and Shalkowski., 2015] Bueno, O. and Shalkowski., S. A. (2015). Empirically grounded philosophical theorizing. In Daly, C., editor, *The Palgrave Handbook of Philosophical Methods*. Palgrave.

[Carnap, 1939] Carnap, R. (1939). Foundation of logic and mathematics. *International Encyclopedia of Unified Science, The University of Chicago Press, Chicago.*, I/3.

[Carnap, 1950] Carnap, R. (1950). Empiricism, semantics, and ontology. *Revue Internationale De Philosophie*, 4(2):20–40.

[Church, 1951] Church, A. (1951). The need for abstract entities. *American Academy of Arts and Sciences Proceedings*, 80:100–113.

[Colyvan, 2001] Colyvan, M. (2001). *The indispensability of Mathematics*. Oxford U. P., New York.

[Enoch and Schechter, 2008] Enoch, D. and Schechter, J. (2008). How are basic belief-forming methods justified? *Philosophy, Philosophy and Phenomenological Research*, 76(3):547–579.

[Field, 1982] Field, H. (1982). Realism and anti-realism about mathematics. *Philosophical Topics*, 13(1):45–69.

[Field, 1989] Field, H. (1989). *Realism, Mathematics & Modality*. Basil Blackwell.

[Laudan, 1981] Laudan, L. (1981). A confutation of convergent realism. *Philosophy of Science*, 48(1):19–49.

[Leng, 2010] Leng, M. (2010). *Mathematics and Reality*. Oup Oxford.

[Maddy, 1992] Maddy, P. (1992). Indispensability and practice. *Journal of Philosophy*, 89(6):275–289.

[Maddy, 1997] Maddy, P. (1997). *Naturalism in Mathematics*. Oxford University Press.

[Maddy, 2011] Maddy, P. (2011). *Defending the Axioms: On the Philosophical Foundations of Set Theory*. Oxford University Press.

[Molinini, 2015] Molinini, D. (2015). Evidence, explanation and enhanced indispensability. *In F. Pataut, D. Molinini, A. Sereni (eds.) Synthese, S.I "Indispensability and explanation", this volume.*

[Panza and Sereni, 2013] Panza, M. and Sereni, A. (2013). *Plato's Problem. An Introduction to Mathematical Platonism*. Palgrave MacMillan, Basingstoke (UK).

[Panza and Sereni, 2015] Panza, M. and Sereni, A. (2015). On the indispensable premises of the indispensability argument. In G. Lolli, M. Panza, G. V., editor, *From Logic to Practice*, pages 241–276. Springer, Dordrecht.

[Peressini, 1997] Peressini, A. (1997). Troubles with indispensability: Applying pure mathematics in physical theoryt. *Philosophia Mathematica*, 5(3):210–227.

[Pincock, 2012] Pincock, C. (2012). Principled limitations on inference to the best explanation. *Manuscipt version presented at the Paris workshop on Indispesability and Expnation, November 19th-20th, 2012*.

[Psillos, 1999] Psillos, S. (1999). *Scientific Realism. How Science Tracks Truths*. Routledge.

[Putnam, 1971] Putnam, H. (1971). *Philosophy of Logic*. Harper and Row, New York.

[Quine, 1960] Quine, W. V. O. (1960). *Word & Object*. The Mit Press.

[Resnik, 1995] Resnik, M. (1995). Scientific vs. mathematical realism: The indispensability argument. *Philosophia Mathematica*, 3, 3th series:166–174.

[Resnik, 1997] Resnik, M. (1997). *Mathematics as a Science of Patterns*. Clarendon Press, Oxford.

[Rizza, 2011] Rizza, D. (2011). Magicicada, mathematical explanation and mathematical realism. *Erkenntnis*, 74(1):101–114.

[Sereni, 2015] Sereni, A. (2015). Equivalent explanations and mathematical realism. reply to " Evidence, explanation, and enhanced indispensability". *In F. Pataut, D. Molinini, A. Sereni (eds.) Synthese, S.I "Indispensability and explanation", this volume.*

[Wagner, 1996] Wagner, S. J. (1996). Prospect for Platonism. In Morton, A. and Stich, S. P., editors, *Benacerraf and his Critics*, pages 73–99. Blackwell, Oxford.